# Linear Algebra MAT 310

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## YouTube lectures

The author of the textbook, Profesor Sheldon Axler
uploaded video of his lectures on YouTube:
https://www.youtube.com/watch?v=5DZV4nsEkNk

I strongly recommend to watch. The video clips are are short.

The reference to the whole list:
https://www.youtube.com/playlist?list=PLGAnmvB9m7zOBVCZBUUmSinFV0wEir2Vw

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## Fields

Reminding:

**Definition: a field**
A *field* is a set equipped with addition and multiplication which are:
**commutative,**
**associative,**
**have identities,**
**additive inverse,**
**multiplicative inverse,**
**distributivity property.**

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/2$.

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## Addition and scalar multiplication in a set

Let $V$ be a set.

**Definition: addition in a set.**
An *addition* in $V$ is a function $V \times V \to V : (u, v) \mapsto u + v$.

Let $\mathbb{F}$ be a field.

**Definition: a scalar multiplication in a set.**
A *scalar multipliation* on $V$ is a function $\mathbb{F} \times V \to V : (\lambda, u) \mapsto \lambda u$.

**Example.** Let $S$ be a set, $\mathbb{F}^S$ denote the set of all maps $S \to \mathbb{F}$.

Addition in $V$:
for $f, g \in \mathbb{F}^S$ define $f + g$ by $(f + g)(x) = f(x) + g(x)$ for $\forall x \in S$.

Scalar multiplication:
for $f \in \mathbb{F}^S$ and $\lambda \in \mathbb{F}$ define $\lambda f$ by $(\lambda f)(x) = \lambda f(x)$ for $\forall x \in S$.



## Vector spaces

Let $\mathbb{F}$ be a field.

**Definition: a vector (or linear) space.**

A *vector space* over $\mathbb{F}$ is a set $V$ equipped
with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$,
$1u = u$ for $\forall u \in V$,
$a(u + v) = au + av$ for $\forall a \in \mathbb{F}$ and $\forall u, v \in V$,
$(a + b)u = au + bu$ for $\forall a, b \in \mathbb{F}$ and $\forall u \in V$.

**Examples.** $\mathbb{F}^n$, $\mathbb{F}^S$, $\mathbb{C}$ is a vector space over $\mathbb{R}$.
We say $\mathbb{C}$ is a *real vector space*. $\mathbb{C}$ is also a *complex vector space*.

What is the smallest vector space over $\mathbb{F}$?

## The zero vector

**Theorem.** In any vector space $V$, $0u = 0$ for every $u \in V$.

what are the zeros?

**Proof.** $0 = 0 + 0$.
Hence $0u = (0+0)u$
$= 0u + 0u$.

Therefore $0u - 0u = 0u + 0u - 0u$, and $0 = 0u$. ∎

**Theorem.** In any vector space $V$, $a0 = 0$ for every $a \in \mathbb{F}$.

**Proof.** $a0 = a(0+0) = \dots$



## The space of polynomials

A polynomial in a variable $x$ over a field $\mathbb{F}$ is an expression

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m \text{, where } a_k \in \mathbb{F}.$$

Polynomials in a variable $X$ over a field $\mathbb{F}$ form a vector space over $\mathbb{F}$.

Notation $\mathbb{F}[x]$. In Axler's book $\mathcal{P}(\mathbb{F})$.

## Linear maps

Let $V$ and $W$ be vector spaces over $\mathbb{F}$.

**Definition: a linear map.**
A map $T : V \to W$ is called *linear*, if

$$T(u_1 + u_2) = Tu_1 + Tu_2 \text{ for } \forall u_1, u_2 \in V \text{ (additivity)}$$
$$T(\lambda u) = \lambda Tu \text{ for } \forall \lambda \in \mathbb{F} \text{ and } \forall u \in V \text{ (homogeneity).}$$

The set of all linear maps $V \to W$ is denoted by $\mathcal{L}(V, W)$.

**Examples:** $0 \in \mathcal{L}(V, W)$.
Identity map $\mathrm{id} \in \mathcal{L}(V, V)$ $\mathrm{id}(u) = u$.
Differentiation $\mathbb{R}[x] \to \mathbb{R}[x] : p(x) \mapsto \frac{dp}{dx}(x)$.
Integration $\mathbb{R}[x] \to \mathbb{R} : p(x) \mapsto \int_0^1 p(x)dx$.

$\mathcal{L}(V, W)$ is a vector space.



## Subspaces

Let $V$ be a vector space over $\mathbb{F}$ and $U \subset V$.

**Definition: subspace.**
$U$ is called a *(vector or linear) subspace* of $V$ if $U$ is a vector space with the same addition and multiplication as on $V$.

A subset $U$ of a vector space $V$ is a subspace iff

$0 \in U$,
$u + v \in U$ if $u, v \in U$,
$\lambda u \in U$ if $\lambda \in \mathbb{F}$ and $u \in U$.

**Examples of subspaces.** In $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$.

Linear conditions: continuity, differentiablity.

## Intersection and sums

**Theorem.** Intersection of any collection of subspaces is a subspace.

**Definition: sum of subsets** Let $U_1, \dots, U_m$ be subsets of a vector space $V$.

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_1 \in U_1, \dots, u_m \in U_m\}$$

**Theorem.** If $U_1, \dots, U_m$ are subspaces of a vector space $V$, then $U_1 + \cdots + U_m$ is the smallest subspace of $V$ containing $U_1, \dots, U_m$.



## Direct sums

**Definition: direct sum**
$U_1 + \cdots + U_m$ is called a *direct sum* and is denoted by $U_1 \oplus \cdots \oplus U_m$ if each $u \in U_1 + \cdots + U_m$ has a unique presentation as $u_1 + \cdots + u_m$ with $u_j \in U_j$.

**Theorem.** Let $U_1, \dots U_m$ be subspaces of $V$. Then $U_1 + \cdots + U_m$ is a direct sum iff there is only one way to represent $0$ as $u_1 + \cdots + u_m$ with $u_j \in U_j$.

Which way?

**Special case:** $m = 2$**.**
If $U, W$ are subspaces of a vector space $V$, then

$$U + W = U \oplus W \quad \text{iff} \quad U \cap W = \{0\}.$$