Linear Algebra

Oleg Viro

01/30/2020

YouTube lectures

The author of the textbook, Profesor Sheldon Axler

YouTube lectures

The author of the textbook, Profesor Sheldon Axler

uploaded video of his lectures on YouTube:

YouTube lectures

The author of the textbook, Profesor Sheldon Axler uploaded video of his lectures on YouTube: https://www.youtube.com/watch?v=5DZV4nsEkNk

YouTube lectures

The author of the textbook, Profesor Sheldon Axler
uploaded video of his lectures on YouTube:
https://www.youtube.com/watch?v=5DZV4nsEkNk

I strongly recommend to watch. The video clips are are short.

YouTube lectures

The author of the textbook, Profesor Sheldon Axler uploaded video of his lectures on YouTube:
https://www.youtube.com/watch?v=5DZV4nsEkNk

I strongly recommend to watch. The video clips are are short.

The reference to the whole list:
https://www.youtube.com/playlist?list=PLGAnmvB9m7zOBVCZBUUmSinFV0wEir2Vw

Fields

Reminding:

Definition: a field

A *field* is a set equipped with

.

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication

.

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication which are:

.

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication which are:

commutative

.

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication which are:

commutative,

associative

.

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication which are:

commutative,

associative,

have identities

.

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication which are:

commutative,

associative,

have identities,

additive inverse

.

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication which are:
commutative,
associative,
have identities,
additive inverse,
multiplicative inverse

.

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication which are:
commutative,
associative,
have identities,
additive inverse,
multiplicative inverse,
distributivity property.

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication which are:
commutative,
associative,
have identities,
additive inverse,
multiplicative inverse,
distributivity property.

Examples: $\mathbb{R}$

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication which are:

commutative,

associative,

have identities,

additive inverse,

multiplicative inverse,

distributivity property.

Examples: $\mathbb{R}$, $\mathbb{C}$

Fields

Reminding:

Definition: a field

A *field* is a set equipped with addition and multiplication which are:
commutative,
associative,
have identities,
additive inverse,
multiplicative inverse,
distributivity property.

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/2$.

Addition and scalar multiplication in a set

Addition and scalar multiplication in a set

Let V be a set.

Addition and scalar multiplication in a set

Let V be a set.

Definition: addition in a set.
An *addition* in V is a function $V \times V \to V : (u, v) \mapsto u + v$.

Addition and scalar multiplication in a set

Let V be a set.

Definition: addition in a set.
An *addition* in V is a function $V \times V \to V : (u, v) \mapsto u + v$.

Let $\mathbb{F}$ be a field.

Addition and scalar multiplication in a set

Let V be a set.

Definition: addition in a set.
An *addition* in V is a function $V \times V \to V : (u, v) \mapsto u + v$.

Let $\mathbb{F}$ be a field.

Definition: a scalar multiplication in a set.
A *scalar multipliation* on V is a function $\mathbb{F} \times V \to V : (\lambda, u) \mapsto \lambda u$.

Addition and scalar multiplication in a set

Let V be a set.

Definition: addition in a set.
An *addition* in V is a function $V \times V \to V : (u, v) \mapsto u + v$.

Let $\mathbb{F}$ be a field.

Definition: a scalar multiplication in a set.
A *scalar multiplication* on V is a function $\mathbb{F} \times V \to V : (\lambda, u) \mapsto \lambda u$.

Example. Let S be a set

Addition and scalar multiplication in a set

Let V be a set.

Definition: addition in a set.
An *addition* in V is a function $V \times V \to V : (u, v) \mapsto u + v$.

Let $\mathbb{F}$ be a field.

Definition: a scalar multiplication in a set.
A *scalar multiplication* on V is a function $\mathbb{F} \times V \to V : (\lambda, u) \mapsto \lambda u$.

Example. Let S be a set, $\mathbb{F}^S$ denote the set of all maps $S \to \mathbb{F}$.

Addition and scalar multiplication in a set

Let V be a set.

Definition: addition in a set.
An *addition* in V is a function $V \times V \to V : (u, v) \mapsto u + v$.

Let $\mathbb{F}$ be a field.

Definition: a scalar multiplication in a set.
A *scalar multipliation* on V is a function $\mathbb{F} \times V \to V : (\lambda, u) \mapsto \lambda u$.

Example. Let S be a set, $\mathbb{F}^S$ denote the set of all maps $S \to \mathbb{F}$.

Addition in V :

for $f, g \in \mathbb{F}^S$ define $f + g$ by

Addition and scalar multiplication in a set

Let V be a set.

Definition: addition in a set.
An *addition* in V is a function $V \times V \to V : (u, v) \mapsto u + v$.

Let $\mathbb{F}$ be a field.

Definition: a scalar multiplication in a set.
A *scalar multipliation* on V is a function $\mathbb{F} \times V \to V : (\lambda, u) \mapsto \lambda u$.

Example. Let S be a set, $\mathbb{F}^S$ denote the set of all maps $S \to \mathbb{F}$.

Addition in V :

for $f, g \in \mathbb{F}^S$ define $f + g$ by $(f + g)(x) = f(x) + g(x)$ for $\forall x \in S$.

Addition and scalar multiplication in a set

Let V be a set.

Definition: addition in a set.
An *addition* in V is a function $V \times V \to V : (u, v) \mapsto u + v$.

Let $\mathbb{F}$ be a field.

Definition: a scalar multiplication in a set.
A *scalar multiplication* on V is a function $\mathbb{F} \times V \to V : (\lambda, u) \mapsto \lambda u$.

Example. Let S be a set, $\mathbb{F}^S$ denote the set of all maps $S \to \mathbb{F}$.

Addition in V :
for $f, g \in \mathbb{F}^S$ define $f + g$ by $(f + g)(x) = f(x) + g(x)$ for $\forall x \in S$.

Scalar multiplication:
for $f \in \mathbb{F}^S$ and $\lambda \in \mathbb{F}$ define λf by

Addition and scalar multiplication in a set

Let V be a set.

Definition: addition in a set.
An *addition* in V is a function $V \times V \to V : (u, v) \mapsto u + v$.

Let $\mathbb{F}$ be a field.

Definition: a scalar multiplication in a set.
A *scalar multiplication* on V is a function $\mathbb{F} \times V \to V : (\lambda, u) \mapsto \lambda u$.

Example. Let S be a set, $\mathbb{F}^S$ denote the set of all maps $S \to \mathbb{F}$.

Addition in V :
for $f, g \in \mathbb{F}^S$ define $f + g$ by $(f + g)(x) = f(x) + g(x)$ for $\forall x \in S$.

Scalar multiplication:
for $f \in \mathbb{F}^S$ and $\lambda \in \mathbb{F}$ define λf by $(\lambda f)(x) = \lambda f(x)$ for $\forall x \in S$.

Vector spaces

Vector spaces

Let $\mathbb{F}$ be a field.

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped with addition and scalar multiplication such that

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped with addition and scalar multiplication such that the addition is *commutative*

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped with addition and scalar multiplication such that the addition is *commutative*, and *associative*

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped with addition and scalar multiplication such that the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped
with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$,
$1u = u$ for $\forall u \in V$

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped
with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$,
$1u = u$ for $\forall u \in V$,
$a(u + v) = au + av$ for $\forall a \in \mathbb{F}$ and $\forall u, v \in V$

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$,
$1u = u$ for $\forall u \in V$,
$a(u + v) = au + av$ for $\forall a \in \mathbb{F}$ and $\forall u, v \in V$,
$(a + b)u = au + bu$ for $\forall a, b \in \mathbb{F}$ and $\forall u \in V$.

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$,
$1u = u$ for $\forall u \in V$,
$a(u + v) = au + av$ for $\forall a \in \mathbb{F}$ and $\forall u, v \in V$,
$(a + b)u = au + bu$ for $\forall a, b \in \mathbb{F}$ and $\forall u \in V$.

Examples. $\mathbb{F}^n$

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$,
$1u = u$ for $\forall u \in V$,
$a(u + v) = au + av$ for $\forall a \in \mathbb{F}$ and $\forall u, v \in V$,
$(a + b)u = au + bu$ for $\forall a, b \in \mathbb{F}$ and $\forall u \in V$.

Examples. $\mathbb{F}^n$, $\mathbb{F}^S$

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped
with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$,
$1u = u$ for $\forall u \in V$,
$a(u + v) = au + av$ for $\forall a \in \mathbb{F}$ and $\forall u, v \in V$,
$(a + b)u = au + bu$ for $\forall a, b \in \mathbb{F}$ and $\forall u \in V$.

Examples. $\mathbb{F}^n$, $\mathbb{F}^S$, $\mathbb{C}$ is a vector space over $\mathbb{R}$.

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$,
$1u = u$ for $\forall u \in V$,
$a(u + v) = au + av$ for $\forall a \in \mathbb{F}$ and $\forall u, v \in V$,
$(a + b)u = au + bu$ for $\forall a, b \in \mathbb{F}$ and $\forall u \in V$.

Examples. $\mathbb{F}^n$, $\mathbb{F}^S$, $\mathbb{C}$ is a vector space over $\mathbb{R}$.
We say $\mathbb{C}$ is a *real vector space*.

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped
with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$,
$1u = u$ for $\forall u \in V$,
$a(u + v) = au + av$ for $\forall a \in \mathbb{F}$ and $\forall u, v \in V$,
$(a + b)u = au + bu$ for $\forall a, b \in \mathbb{F}$ and $\forall u \in V$.

Examples. $\mathbb{F}^n$, $\mathbb{F}^S$, $\mathbb{C}$ is a vector space over $\mathbb{R}$.
We say $\mathbb{C}$ is a *real vector space*. $\mathbb{C}$ is also a *complex vector space.*

Vector spaces

Let $\mathbb{F}$ be a field.

Definition: a vector (or linear) space.

A *vector space* over $\mathbb{F}$ is a set V equipped with addition and scalar multiplication such that
the addition is *commutative*,
and *associative*,
has *zero* $0 \in V$ such that $0 + u = u$ for $\forall u \in V$,
each element $u \in V$ has *additive inverse* $-u$,
$1u = u$ for $\forall u \in V$,
$a(u + v) = au + av$ for $\forall a \in \mathbb{F}$ and $\forall u, v \in V$,
$(a + b)u = au + bu$ for $\forall a, b \in \mathbb{F}$ and $\forall u \in V$.

Examples. $\mathbb{F}^n$, $\mathbb{F}^S$, $\mathbb{C}$ is a vector space over $\mathbb{R}$.
We say $\mathbb{C}$ is a *real vector space*. $\mathbb{C}$ is also a *complex vector space.*

What is the smallest vector space over $\mathbb{F}$?

The zero vector

Theorem. In any vector space V, $0u = 0$ for every $u \in V$.

The zero vector

Theorem. In any vector space V, $0u = 0$ for every $u \in V$.

what are the zeros?

The zero vector

Theorem. In any vector space V, $0u = 0$ for every $u \in V$.

what are the zeros?

Proof.

The zero vector

Theorem. In any vector space V, $0u = 0$ for every $u \in V$.

what are the zeros?

Proof. $0 = 0 + 0$.

The zero vector

Theorem. In any vector space V, $0u = 0$ for every $u \in V$.

what are the zeros?

Proof. $0 = 0 + 0$.

Hence $$\begin{aligned} 0u &= (0+0)u \\ &= 0u + 0u. \end{aligned}$$

The zero vector

Theorem. In any vector space V, $0u = 0$ for every $u \in V$.

what are the zeros?

Proof. $0 = 0 + 0$.

Hence $0u = (0+0)u$

$= 0u + 0u$.

Therefore $0u - 0u = 0u + 0u - 0u$

The zero vector

Theorem. In any vector space V, $0u = 0$ for every $u \in V$.

what are the zeros?

Proof. $0 = 0 + 0$.

Hence
$$\begin{aligned} 0u &= (0+0)u \\ &= 0u + 0u . \end{aligned}$$

Therefore $0u - 0u = 0u + 0u - 0u$, and $0 = 0u$. ∎

The zero vector

Theorem. In any vector space V, $0u = 0$ for every $u \in V$.

what are the zeros?

Proof. $0 = 0 + 0$.
Hence $0u = (0+0)u$
$= 0u + 0u$.

Therefore $0u - 0u = 0u + 0u - 0u$, and $0 = 0u$. ■

Theorem. In any vector space V, $a0 = 0$ for every $a \in \mathbb{F}$.

The zero vector

Theorem. In any vector space V, $0u = 0$ for every $u \in V$.

what are the zeros?

Proof. $0 = 0 + 0$.

Hence $$\begin{aligned} 0u &= (0+0)u \\ &= 0u + 0u \,. \end{aligned}$$

Therefore $0u - 0u = 0u + 0u - 0u$, and $0 = 0u$. ∎

Theorem. In any vector space V, $a0 = 0$ for every $a \in \mathbb{F}$.

Proof. $a0 = a(0+0) = \dots$

The space of polynomials

The space of polynomials

A polynomial in a variable x over a field $\mathbb{F}$ is an expression

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m \text{, where } a_k \in \mathbb{F}.$$

The space of polynomials

A polynomial in a variable x over a field $\mathbb{F}$ is an expression

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m, \text{ where } a_k \in \mathbb{F}.$$

Polynomials in a variable X over a field $\mathbb{F}$ form a vector space over $\mathbb{F}$.

The space of polynomials

A polynomial in a variable x over a field $\mathbb{F}$ is an expression

$$a_0 + a_1x + a_2x^2 + \cdots + a_mx^m, \text{ where } a_k \in \mathbb{F}.$$

Polynomials in a variable X over a field $\mathbb{F}$ form a vector space over $\mathbb{F}$.

Notation $\mathbb{F}[x]$.

The space of polynomials

A polynomial in a variable x over a field $\mathbb{F}$ is an expression

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m \text{, where } a_k \in \mathbb{F} .$$

Polynomials in a variable X over a field $\mathbb{F}$ form a vector space over $\mathbb{F}$.

Notation $\mathbb{F}[x]$. In Axler's book $\mathcal{P}(\mathbb{F})$.

Linear maps

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.

A map $T : V \to W$ is called *linear*, if

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.

A map $T : V \to W$ is called *linear*, if

$$T(u_1 + u_2) = Tu_1 + Tu_2 \text{ for } \forall u_1, u_2 \in V$$

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.

A map $T : V \to W$ is called *linear*, if

$$T(u_1 + u_2) = Tu_1 + Tu_2 \text{ for } \forall u_1, u_2 \in V \quad \text{(additivity)}$$

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.

A map $T : V \to W$ is called *linear*, if

$T(u_1 + u_2) = Tu_1 + Tu_2$ for $\forall u_1, u_2 \in V$ (additivity)

$T(\lambda u) = \lambda Tu$ for $\forall \lambda \in \mathbb{F}$ and $\forall u \in V$

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.

A map $T : V \to W$ is called *linear*, if

$T(u_1 + u_2) = Tu_1 + Tu_2$ for $\forall u_1, u_2 \in V$ (additivity)

$T(\lambda u) = \lambda Tu$ for $\forall \lambda \in \mathbb{F}$ and $\forall u \in V$ (homogeneity).

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.

A map $T : V \to W$ is called *linear*, if

$T(u_1 + u_2) = Tu_1 + Tu_2$ for $\forall u_1, u_2 \in V$ (additivity)

$T(\lambda u) = \lambda Tu$ for $\forall \lambda \in \mathbb{F}$ and $\forall u \in V$ (homogeneity).

The set of all linear maps $V \to W$ is denoted by $\mathcal{L}(V, W)$.

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.

A map $T : V \to W$ is called *linear*, if

$T(u_1 + u_2) = Tu_1 + Tu_2$ for $\forall u_1, u_2 \in V$ (additivity)

$T(\lambda u) = \lambda Tu$ for $\forall \lambda \in \mathbb{F}$ and $\forall u \in V$ (homogeneity).

The set of all linear maps $V \to W$ is denoted by $\mathcal{L}(V, W)$.

Examples:

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.
A map $T : V \to W$ is called *linear*, if

$T(u_1 + u_2) = Tu_1 + Tu_2$ for $\forall u_1, u_2 \in V$ (additivity)
$T(\lambda u) = \lambda T u$ for $\forall \lambda \in \mathbb{F}$ and $\forall u \in V$ (homogeneity).

The set of all linear maps $V \to W$ is denoted by $\mathcal{L}(V, W)$.

Examples: $0 \in \mathcal{L}(V, W)$.

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.
A map $T : V \to W$ is called *linear*, if

$T(u_1 + u_2) = Tu_1 + Tu_2$ for $\forall u_1, u_2 \in V$ (additivity)
$T(\lambda u) = \lambda Tu$ for $\forall \lambda \in \mathbb{F}$ and $\forall u \in V$ (homogeneity).

The set of all linear maps $V \to W$ is denoted by $\mathcal{L}(V, W)$.

Examples: $0 \in \mathcal{L}(V, W)$.
Identity map $\mathrm{id} \in \mathcal{L}(V, V)$

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.
A map $T : V \to W$ is called *linear*, if

$T(u_1 + u_2) = Tu_1 + Tu_2$ for $\forall u_1, u_2 \in V$ (additivity)

$T(\lambda u) = \lambda Tu$ for $\forall \lambda \in \mathbb{F}$ and $\forall u \in V$ (homogeneity).

The set of all linear maps $V \to W$ is denoted by $\mathcal{L}(V, W)$.

Examples: $0 \in \mathcal{L}(V, W)$.
Identity map $\mathrm{id} \in \mathcal{L}(V, V)$ $\mathrm{id}(u) = u$.

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.
A map $T : V \to W$ is called *linear*, if

$$T(u_1 + u_2) = Tu_1 + Tu_2 \text{ for } \forall u_1, u_2 \in V \text{ (additivity)}$$
$$T(\lambda u) = \lambda Tu \text{ for } \forall \lambda \in \mathbb{F} \text{ and } \forall u \in V \text{ (homogeneity).}$$

The set of all linear maps $V \to W$ is denoted by $\mathcal{L}(V, W)$.

Examples: $0 \in \mathcal{L}(V, W)$.
Identity map $\mathrm{id} \in \mathcal{L}(V, V)$ $\mathrm{id}(u) = u$.
Differentiation $\mathbb{R}[x] \to \mathbb{R}[x] : p(x) \mapsto \frac{dp}{dx}(x)$.

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.
A map $T : V \to W$ is called *linear*, if

$$T(u_1 + u_2) = Tu_1 + Tu_2 \text{ for } \forall u_1, u_2 \in V \text{ (additivity)}$$
$$T(\lambda u) = \lambda Tu \text{ for } \forall \lambda \in \mathbb{F} \text{ and } \forall u \in V \text{ (homogeneity).}$$

The set of all linear maps $V \to W$ is denoted by $\mathcal{L}(V, W)$.

Examples: $0 \in \mathcal{L}(V, W)$.
Identity map $\mathrm{id} \in \mathcal{L}(V, V)$ $\mathrm{id}(u) = u$.
Differentiation $\mathbb{R}[x] \to \mathbb{R}[x] : p(x) \mapsto \frac{dp}{dx}(x)$.
Integration $\mathbb{R}[x] \to \mathbb{R} : p(x) \mapsto \int_0^1 p(x)dx$.

Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

Definition: a linear map.
A map $T : V \to W$ is called *linear*, if

$T(u_1 + u_2) = Tu_1 + Tu_2$ for $\forall u_1, u_2 \in V$ (additivity)
$T(\lambda u) = \lambda Tu$ for $\forall \lambda \in \mathbb{F}$ and $\forall u \in V$ (homogeneity).

The set of all linear maps $V \to W$ is denoted by $\mathcal{L}(V, W)$.

Examples: $0 \in \mathcal{L}(V, W)$.
Identity map $\mathrm{id} \in \mathcal{L}(V, V)$ $\mathrm{id}(u) = u$.
Differentiation $\mathbb{R}[x] \to \mathbb{R}[x] : p(x) \mapsto \frac{dp}{dx}(x)$.
Integration $\mathbb{R}[x] \to \mathbb{R} : p(x) \mapsto \int_0^1 p(x)dx$.

$\mathcal{L}(V, W)$ is a vector space.

Subspaces

Subspaces

Let V be a vector space over $\mathbb{F}$ and $U \subset V$.

Subspaces

Let V be a vector space over $\mathbb{F}$ and $U \subset V$.

Definition: subspace.
U is called a *(vector or linear) subspace* of V if U is a vector space with the same addition and multiplication as on V.

Subspaces

Let V be a vector space over $\mathbb{F}$ and $U \subset V$.

Definition: subspace.
U is called a *(vector or linear) subspace* of V if U is a vector space with the same addition and multiplication as on V.

A subset U of a vector space V is a subspace iff

Subspaces

Let V be a vector space over $\mathbb{F}$ and $U \subset V$.

Definition: subspace.
U is called a *(vector or linear) subspace* of V if U is a vector space with the same addition and multiplication as on V .

A subset U of a vector space V is a subspace iff

$0 \in U$,

$u + v \in U$ if $u, v \in U$,

$\lambda u \in U$ if $\lambda \in \mathbb{F}$ and $u \in U$.

Subspaces

Let V be a vector space over $\mathbb{F}$ and $U \subset V$.

Definition: subspace.
U is called a *(vector or linear) subspace* of V if U is a vector space with the same addition and multiplication as on V.

A subset U of a vector space V is a subspace iff

$0 \in U$,

$u + v \in U$ if $u, v \in U$,

$\lambda u \in U$ if $\lambda \in \mathbb{F}$ and $u \in U$.

Examples of subspaces.

Subspaces

Let V be a vector space over $\mathbb{F}$ and $U \subset V$.

Definition: subspace.
U is called a *(vector or linear) subspace* of V if U is a vector space with the same addition and multiplication as on V.

A subset U of a vector space V is a subspace iff

$0 \in U$,
$u + v \in U$ if $u, v \in U$,
$\lambda u \in U$ if $\lambda \in \mathbb{F}$ and $u \in U$.

Examples of subspaces. In $\mathbb{R}^1$

Subspaces

Let V be a vector space over $\mathbb{F}$ and $U \subset V$.

Definition: subspace.
U is called a *(vector or linear) subspace* of V if U is a vector space with the same addition and multiplication as on V .

A subset U of a vector space V is a subspace iff

$0 \in U$,

$u + v \in U$ if $u, v \in U$,

$\lambda u \in U$ if $\lambda \in \mathbb{F}$ and $u \in U$.

Examples of subspaces. In $\mathbb{R}^1$, $\mathbb{R}^2$

Subspaces

Let V be a vector space over $\mathbb{F}$ and $U \subset V$.

Definition: subspace.
U is called a *(vector or linear) subspace* of V if U is a vector space with the same addition and multiplication as on V.

A subset U of a vector space V is a subspace iff

$$0 \in U,$$
$$u + v \in U \text{ if } u, v \in U,$$
$$\lambda u \in U \text{ if } \lambda \in \mathbb{F} \text{ and } u \in U.$$

Examples of subspaces. In $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$.

Subspaces

Let V be a vector space over $\mathbb{F}$ and $U \subset V$.

Definition: subspace.
U is called a *(vector or linear) subspace* of V if U is a vector space with the same addition and multiplication as on V.

A subset U of a vector space V is a subspace iff

$0 \in U$,

$u + v \in U$ if $u, v \in U$,

$\lambda u \in U$ if $\lambda \in \mathbb{F}$ and $u \in U$.

Examples of subspaces. In $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$.

Linear conditions: continuity

Subspaces

Let V be a vector space over $\mathbb{F}$ and $U \subset V$.

Definition: subspace.
U is called a *(vector or linear) subspace* of V if U is a vector space with the same addition and multiplication as on V.

A subset U of a vector space V is a subspace iff

$0 \in U$,

$u + v \in U$ if $u, v \in U$,

$\lambda u \in U$ if $\lambda \in \mathbb{F}$ and $u \in U$.

Examples of subspaces. In $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$.

Linear conditions: continuity, differentiablity.

Intersection and sums

Intersection and sums

Theorem. Intersection of any collection of subspaces is a subspace.

Intersection and sums

Theorem. Intersection of any collection of subspaces is a subspace.

Definition: sum of subsets Let $U_1, \ldots, U_m$ be subsets of a vector space V.

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_1 \in U_1, \ldots, u_m \in U_m\}$$

Intersection and sums

Theorem. Intersection of any collection of subspaces is a subspace.

Definition: sum of subsets Let $U_1, \ldots, U_m$ be subsets of a vector space V .

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_1 \in U_1, \ldots, u_m \in U_m\}$$

Theorem. If $U_1, \ldots, U_m$ are subspaces of a vector space V , then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Direct sums

Direct sums

Definition: direct sum

$U_1 + \cdots + U_m$ is called a *direct sum* and is denoted by $U_1 \oplus \cdots \oplus U_m$ if each $u \in U_1 + \cdots + U_m$ has a unique presentation as $u_1 + \cdots + u_m$ with $u_j \in U_j$.

Direct sums

Definition: direct sum

$U_1 + \cdots + U_m$ is called a *direct sum* and is denoted by $U_1 \oplus \cdots \oplus U_m$ if each $u \in U_1 + \cdots + U_m$ has a unique presentation as $u_1 + \cdots + u_m$ with $u_j \in U_j$.

Theorem. Let $U_1, \dots U_m$ be subspaces of V . Then $U_1 + \cdots + U_m$ is a direct sum iff there is only one way to represent 0 as $u_1 + \cdots + u_m$ with $u_j \in U_j$.

Direct sums

Definition: direct sum
$U_1 + \cdots + U_m$ is called a *direct sum* and is denoted by $U_1 \oplus \cdots \oplus U_m$ if each $u \in U_1 + \cdots + U_m$ has a unique presentation as $u_1 + \cdots + u_m$ with $u_j \in U_j$.

Theorem. Let $U_1, \ldots U_m$ be subspaces of V . Then $U_1 + \cdots + U_m$ is a direct sum iff there is only one way to represent 0 as $u_1 + \cdots + u_m$ with $u_j \in U_j$.

Which way?

Direct sums

Definition: direct sum

$U_1 + \cdots + U_m$ is called a *direct sum* and is denoted by $U_1 \oplus \cdots \oplus U_m$ if each $u \in U_1 + \cdots + U_m$ has a unique presentation as $u_1 + \cdots + u_m$ with $u_j \in U_j$.

Theorem. Let $U_1, \ldots U_m$ be subspaces of V . Then $U_1 + \cdots + U_m$ is a direct sum iff there is only one way to represent 0 as $u_1 + \cdots + u_m$ with $u_j \in U_j$.

Which way?

Special case: $m = 2$ **.**

If U, W are subspaces of a vector space V , then

$$U + W = U \oplus W \quad \text{iff} \quad U \cap W = \{0\} .$$