Lie Groups

G - a group which is a manifold s.t. the map

\[ G \times G \ni (a, b) \rightarrow a^{-1} e G \]

is differentiable

is a Lie Group

Left translations: \( \forall a \in G : L_a : G \rightarrow G \)

\[ L_a(b) = a \cdot b \]

Right translations: \( \forall a \in G : R_a : G \rightarrow G \)

\[ R_a(b) = b \cdot a \]

Def A Riemannian metric \( g \) on \( G \) is

left invariant iff

\[ g_b(X_b, Y_b) = g_a(L_aX_b, L_aY_b) \]

\( \forall a, b \in G \)

\( \forall X_b, Y_b \in T_b G \)

(similarly right invariant)

\( g \) is bi-invariant iff it is left and right invariant.

Def

A vector field \( X \) on \( G \) is left invariant iff

\( \forall a \in G : L_a X = X \) \( (L_a^* X_b = X_{a b} \forall a, b \in G) \)

(note that since \( L_a \) is diffeomorphism we can pushforward vector fields!)

Left invariant vector fields are completely determined by their values at \( e \) - identity element in \( G \).
This enables to introduce additional structure in $T_eG$.

Take any vector $X_e \in T_eG$.

Define a left invariant vector field $X$ by

$$X_a = L_a * X_e$$

Taking another $Y_e \in T_eG$ we have also $Y$ s.t.

$$Y_a = L_a * X_e$$

We equip $T_eG$ with a structure of Lie algebra by setting

$$[X_e, Y_e] = [X, Y]_e.$$

This is ok, since $L_a * [X, Y] = [L_a * X, L_a * Y]$.

Exercise: check that

$$\text{Ad}_a [X, Y] = [\text{Ad}_a X, \text{Ad}_a Y]$$

How to define a left invariant metric on $G$?

Take any scalar product $\langle \cdot, \cdot \rangle_e$ in $T_eG$.

Define

$$(L1) \quad g_a(X_a, Y_a) = \left\langle L_a^{-1} * X_a, L_a^{-1} * Y_a \right\rangle_e \quad \forall a \in G, \forall X_a \in T_aG$$

This is clearly left invariant because:

$$L_a \circ L_b = L_{ab}, \quad L_c \circ L_a \circ L_b = L_{ca \circ b}$$

In the same way we define right invariant metric on $G$.

This see exercise 7 p. 46 in Do Carmo.

Let $G$ be a compact connected Lie group $G$.

Then $G$ admits a bi-invariant Riemannian metric.
\[ L_{c} \circ L_{a} = L_{ca} \]

\[ L_{c_{ab}} \circ L_{a_{b}^{*}} = L_{ca_{b}^{*}} \]

\[ g_{ab} \left( L_{a_{b}^{*}} X_{b}, L_{a_{b}^{*}} Y_{b} \right) = \]

\[ = \left< L_{a_{b}^{-1} a_{b}^{*}} L_{a_{b}^{*}} X_{b}, L_{a_{b}^{-1} a_{b}^{*}} L_{a_{b}^{*}} Y_{b} \right>_{e} = \]

\[ = \left< L_{(a_{b}^{-1} a_{b}^{*}) a_{b}^{*}} L_{a_{b}^{*}} X_{b}, L_{(a_{b}^{-1} a_{b}^{*}) a_{b}^{*}} L_{a_{b}^{*}} Y_{b} \right>_{e} = \]

\[ = \left< L_{a_{b}^{-1} a_{b}^{*}} X_{b}, L_{a_{b}^{-1} a_{b}^{*}} Y_{b} \right>_{e} = g_{a_{b}} (X_{b}, Y_{b}) , \]
Example: Upper half plane

\[ \mathcal{H}_+ = \{ \mathbb{R}^2 \setminus (x, y) : y > 0 \} \]

Group structure: on the space of functions \( f(x, y) : \mathbb{R} \rightarrow \mathbb{R} \)

s.t. \( f(x, y)(t) = yt + x \) consider composition:

\[ f(x', y') = f(x, y) \]

\[ f(x', y') \circ f(x, y)(t) = f(x', y')(f(x, y)(t)) = f(x', y)(yt + x) = \]

\[ = y'yt + y'x + x' = \]

\[ = f(y'x + x', y'y)(t) \]

\[ (x', y') \cdot (x, y) = (y'x + x', y'y) \in \mathcal{H}_+ \]

Lie Group with \( e = (0, 1) \) and inverse \( (x, y)^{-1} = (-\frac{x}{y}, \frac{1}{y}) \)

Left invariant vector fields

Take \( \partial_x \) at \( e \), \( L_{(a, b)}(x, y) = (ax + a, by) \)

\[ L_{(a, b)}* \partial_x |_{(e, e)} = b \partial_x |_{(a, b)} \]

\[ L_{(a, b)}* \partial_x |_{(e, e)} = (b, 0) \]

\[ X_1 = y \partial_x \]

\[ X_2 = y \partial_y \]

\[ [X_1, X_2] = -y \partial_x = -X_1 \]

Right invariant vector fields

\[ R_{(a, b)}(x, y) = (x, y) \cdot (a, b) = (ya + x, yb) \]

\[ \partial_x \mapsto \gamma(t) = (t, 1) \mapsto \partial_x \]

\[ \partial_y \mapsto (0, t) \mapsto R(0, b + t) = ((e(t), b(t + 1))) \]

\[ Y_1 = \partial_x \]

\[ Y_2 = x \partial_x + y \partial_y \]

\[ [Y_1, Y_2] = Y_1 \]
left invariant metric which is $\delta_{ij}$ at $(0,1)$

\[
\begin{align*}
g(x, y) &= \frac{1}{y} \cdot \frac{1}{y} \langle x, y \rangle e = \frac{1}{y^2} \\
g(x, y) &= \frac{1}{y} \cdot \frac{1}{y} \langle y, y \rangle e = 0 \\
g(y, y) &= \frac{1}{y} \cdot \frac{1}{y} \langle y, y \rangle e = 0 \\
\end{align*}
\]

\[
g = \frac{dx^2 + dy^2}{y^2}
\]

right invariant metric which is $\delta_{ij}$ at $(0,1)$

\[
\begin{align*}
g(x, x) &= \langle x, x \rangle e = 1 \\
g(x, y) &= \langle -\frac{x}{y} x + \frac{1}{y} y, x \rangle e = -\frac{x}{y} \\
g(y, y) &= \langle -\frac{x}{y} y + \frac{1}{y} y, \frac{x}{y} y + \frac{1}{y} y \rangle e = \frac{x^2}{y^2} + \frac{1}{y^2} \\
\end{align*}
\]

\[
g = dx^2 - 2 \frac{x}{y} dx dy + \frac{x^2 + 1}{y^2} dy^2 = \\
= (dx - \frac{x}{y} dy)^2 + \frac{1}{y^2} dy^2 = \\
= \frac{(ydx - xdy)^2 + dy^2}{y^2} = d(\frac{x}{y})^2 + (\frac{dy}{y})^2
\]

\[
(\begin{pmatrix}
x^1 = \frac{x}{y} \\
y^1 = \log y
\end{pmatrix})
\]

\[
(\begin{pmatrix}
x^0 = \frac{x}{y} \\
y^0 = \log y
\end{pmatrix})
\]
If $g$ is a bi-invariant metric on $G$ then the scalar product $\langle \cdot, \cdot \rangle$ induced by $g$ in $T_e G$ satisfies

$$\langle [U, X], Y \rangle + \langle X, [U, Y] \rangle = 0. \quad (\ast)$$

And the opposite is true:

If we have a scalar product in $T_e G$ such that $(\ast)$ holds then the metric defined by (LI) is bi-invariant on $G$. (proof do Carmo p. 40-41)

A) Product metric

$(M_1, g_1), (M_2, g_2)$

Consider $M_1 \times M_2$ with projections $\pi_1 : M_1 \times M_2 \to M_1$

$$\pi_2 : M_1 \times M_2 \to M_2$$

$$g_1 \oplus g_2 (X, Y) = g_1 (\pi_1 X, \pi_1 Y) + g_2 (\pi_2 X, \pi_2 Y)$$

e.g. take a torus

$$T^n = S^1 \times \ldots \times S^1$$

and take a metric $g$ on $S^1$ as an induced metric that $S^1$ gets from euclidean metric in $\mathbb{R}^2$.

$$\implies g_{T^n} = g \oplus \ldots \oplus g$$

$n$-times.

Flat torus
5) Every manifold (Hausdorff + countable basis) admits a Riemannian metric.

Partition of unity:

family of functions \( f_\alpha : M \rightarrow \mathbb{R} \) s.t.

1) \( \forall \alpha \, \, f_\alpha \geq 0 \) and closure of \( \text{supp } f_\alpha \subset \bigcup U_\alpha \)

2) \( \{ U_\alpha \} \) is a locally finite cover of \( M \) i.e.

\[ \bigcup U_\alpha = M, \quad \text{and} \quad \forall p \in M \, \exists \, \text{finite number of } \alpha \text{ for which } U_\alpha \ni p \]

3) \( \sum_\alpha f_\alpha (p) = 1 \) \( \forall p \in M \)

Partition of unity always exists on \( M \) which is Hausdorff and has countable basis (see do Carmo p. 30)

\[ \Rightarrow \text{take such a partition on } M \]

\( \{ f_\alpha \}, \, \{ U_\alpha \} \quad \text{coordinate charts} \)

In each \( U_\alpha \) we define a metric \( g \) s.t. in coordinate basis

\[ g^\alpha (\partial / \partial x^\mu, \partial / \partial x^\nu) = \delta^\mu_\nu \]

\[ \Rightarrow g = \sum_\alpha f_\alpha (p) g^\alpha. \]