REMARKS on terminology

\[ \Sigma^n \subset (M^{n+k}, g) \]

\((e^a_x, n_a)\) - orthonormal frame on \(\Sigma^n\)

\[ de^a_\mu = b^a_{\mu \alpha} e^\alpha + Y^a_\mu e^a_{\tau} \]

\[ b^a_{\mu \alpha} = b^a_{\mu \tau} \theta^\tau \theta^\alpha \]

\[ B^a = b^a_{\mu \alpha} \theta^\mu \theta^\alpha \cdot n_a \]

second fundamental form.

\(n \in (T\Sigma)^\perp, \quad x, y \in T\Sigma \quad x = x^a e^a, \quad y = y^a e^a \quad n = n_a n_a \)

\[ H^a(x, y) = g(B(x, y), n) = g_{ab} B^a(x, y) n^b = \]

\[ = g_{ab} b^a_{\mu \alpha} x^\mu y^\alpha n^a n_b = \]

\[ = b^a_{\mu \alpha} x^\mu y^\alpha n_a = \]

\[ = (b^a_{\mu \alpha} y^\alpha n_a) x^\mu \]

Observe that \(S_\eta\) is defined such that by\( \eta\):

\[ S_\eta(x) : TM \to TM \]

Define \(S_\eta : TM \to TM\) by:

\[ [S_\eta(x)]_\tau = b^a_{\mu \alpha} x^\mu n_a e^\tau \]

\[ g(S_\eta(x), y) = H^a(x, y) = g(S_\eta(y), x) \]
$S_n$ is a symmetric operator on $T_p \mathbb{M}$ for each $p \in \mathbb{Z}$.

There exists an orthonormal basis in $T_p \mathbb{M}$ s.t.

$$S_n(e^\mu) = \lambda^\mu e^\mu$$

where $\lambda^\mu$ are real eigenvalues,

no summation.

$(e^\mu, n^\alpha)$ and $e^\mu$ diagonalize $S_n$.

If $|\eta|=1$ and $\Sigma^n$ is hypersurface ($n=1$) then we can take $(e^\mu, \eta)$ and $\eta$ is unique if we want $(e^\mu, \eta)$ to agree with the orientation of $T_p \mathbb{M}$.

$e^\mu$ - are called principal directions

$\lambda^\mu$ - are called principal curvatures

$\det S_n = \lambda_1 \cdots \lambda_n$ Gauss-Kronecker curvature

$$\frac{1}{n} \text{Tr} S_n = \frac{\lambda_1 + \cdots + \lambda_n}{n}$$

mean curvature

It's easy to see that if $\lambda^\mu$ are distinct then

$$\lambda^\mu$$
Isoparametric hypersurfaces

\[ \Sigma^n \subset (M^{n+1}, g), \text{ where } (M^{n+1}, g) \text{ is a space of constant curvature, is called ISOPARAMETRIC iff all its principal curvatures are constant.} \]

Results:

1) \( \left[ M^{n+1} = \mathbb{R}^{n+1} \right] \Rightarrow \Sigma \text{ has at most } [ \text{two} ] \text{ distinct principal curvatures} \)

and must be an open subset of

or

a) hyperplane

or

b) hypersphere

c) spherical cylinder \( S^k \times \mathbb{R}^{n-k} \)

Leray-Civita for \( n+1 = 3 \) 1937
Segre for arbitrary \( n \) 1938

2) \( M^{n+1} = \mathbb{H}^{n+1} \Rightarrow \) isoparametric \( \Sigma \) has at most 2 distinct principal curvatures

and must be either

- open subset of \( S^k \times \mathbb{H}^{n-k} \) or

- be totally umbilic. Cartan 1938

3) \( M^{n+1} = S^{n+1} \) more interesting situation!

Cartan 1938 found isoparametric \( \Sigma^n \subset S^{n+1} \)

with 1, 2, 3 and 4 distinct principal curvatures.

Münzner: number \( g \) of distinct principal curvatures of an isoparametric hypersurface \( \Sigma^n \subset S^{n+1} \) can be 1, 2, 3, 4 or 6.
\( g \leq 3 \)

**Cartan:**

\( g = 1 \Rightarrow \Sigma \) is a great or small sphere in \( \mathbb{S}^{n+1} \)

\( g = 2 \Rightarrow \Sigma \) is a standard product of two spheres

\( \mathbb{S}^k(r) \times \mathbb{S}^{n-k}(s) \subset \mathbb{S}^{n+1} \)

\( g = 3 \Rightarrow \) all the principal curvatures have to have the same multiplicity 1, 2, 4, or 8.

\( g = 6 \Rightarrow \) all have the same multiplicity, \( m = 1 \) or 2.

**Cartan/Münzner:**

Isoparametric hypersurface in \( \mathbb{S}^{n+1} \) is given in terms of a level surface of a polynomial

\[ F : \mathbb{R}^{n+2} \rightarrow \mathbb{R} \]

of degree \( g \) satisfying

\[ |\nabla F|^2 = g^2 (x_1^2 + \ldots + x_{n+2}^2)^{g-1} \]

\[ \Delta F = \frac{m_2 - m_1}{2} g^2 (x_1^2 + \ldots + x_{n+2}^2)^{g-1} \]

where \( m_1 \) and \( m_2 \) are multiplicities of principal curvatures, which either are all equal or there are only two different multiplicities.

Then \( \Sigma^n = \{ x \in \mathbb{R}^{n+2} \text{ s.t. } F = \text{const} \} \)

\[ x_1^2 + \ldots + x_{n+2}^2 = 1 \]
\[
\text{Carlan}
\]
\[ g=3 \implies \begin{cases}
\|\nabla F\|^2 = g \left( x_1^2 + x_2^2 + \cdots + x_n^2 \right)^2, \quad m_1 = m_2 \\
\Delta F = 0
\end{cases}
\]

\[ F = F_{\mu \nu \rho} x^\mu x^\nu x^\rho \]

\[ \nabla_\mu F = 3 F_{\mu \nu \rho} x^\nu x^\rho \]

\[ \|\nabla F\|^2 = g F_{\mu \nu \rho} x^\nu x^\rho F_{\alpha \beta \gamma} x^\alpha x^\beta \]

\[ g g_{\nu \rho} x^\nu x^\rho g_{\alpha \beta} x^\alpha x^\beta \]

\[ \Rightarrow g^\mu_\nu F_{\mu (\nu} F_{\rho)\delta} = g(g_{\nu \rho} g_{\alpha \beta}) \quad 1) \]

\[ g^\mu_\nu F_{\mu \nu} = 0 \quad 2) \]

What are the dimensions \( n+2 \) in which a symmetric tensor with properties 1) and 2) exist?

\[ \text{Carlan} \]

\[ n+2 = 5, 8, 14, 26. \]

\[ \text{dim 5:} \quad A \in M_{3x3}(\mathbb{R}) \text{ s.t. } A^T = A, \quad \text{Tr} A = 0 \]

Space of such matrices is a 5-dim. vector space \( \mathbb{R}^5 \)

\[ A = \begin{pmatrix}
\frac{\alpha^5 - 15 x^4}{10} & \frac{15 x^3}{2} & \frac{15 x^2}{2} \\
-\frac{x^5}{10} & x^4 & 0 \\
0 & \frac{x^5}{10} & -2x^5
\end{pmatrix} \Rightarrow F = \frac{1}{2} \det A \]

Statistics 1) 2) with \( g=3 \).
Why $5, 8, 14, 26$?

Because

$K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

Take $A \in M_{3 \times 3}(K)$ s.t. $A^+ = A$, $\text{Tr} A = 0$

$$n = 2 + 3 \cdot \left(\frac{1}{2}\right)$$

$$F = \frac{1}{2} \det A$$

**Problem** define $\det$ for $A \in M_{3 \times 3}(\mathbb{H})$

$$n = 6$$

**Define**: $G \subset SO(n)$ by

$$G \ni a \iff F(ax, ax, ax) = F(x, x, x).$$

Check that

$$G = \text{SO}(3), \text{SU}(3), \text{Sp}(3), \text{Sp}(4)$$

$$n = 5, 8, 14, 26$$

Each group being in a dimensional irreducible representation

This in particular means that

$\text{SO}(3)$ seats in a nonstandard way in $\text{SO}(5)$