Research Statement

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I am interested in the geometric properties of Gromov-Witten invariants in symplectic topology and algebraic geometry. In particular, my research involves the study of moduli spaces of genus 2 pseudo-holomorphic maps and of real positive-genus Gromov-Witten invariants via analytic methods.

The theory of complex GW-invariants arises from Gromov’s work [Gr] on pseudo-holomorphic curves and Witten’s work [Wi] on σ-models in physics and plays a prominent role in symplectic topology and algebraic geometry. These invariants count pseudo-holomorphic maps from closed (possibly nodal) Riemann surfaces to a symplectic manifold $X$; they are often related to integer counts of curves in $X$. The main object of interest in this theory is the moduli space $\mathcal{M}$ of stable pseudo-holomorphic maps, which is often singular and is stratified by smooth orbifolds. The subspace $\mathcal{M}^0$ of maps from smooth domains is of special interest because it corresponds to irreducible (and often smooth) curves in $X$. Even though $\mathcal{M}$ is conventionally referred to as Gromov’s compactification of $\mathcal{M}^0$, in general $\mathcal{M}^0$ is not dense in $\mathcal{M}$. In the genus 0 case, Gromov’s compactification is in some sense sharp. In the genus 1 case, a sharp compactification is provided by the natural closed subspace $\mathcal{M}^0$ of $\mathcal{M}$ described in [Z1]. This has led to the only proof [Z3] so far of the prediction of [BCOV] for genus 1 GW-invariants of the quintic 3-fold, along with other applications in algebraic geometry. In [N], I construct an analogous sharp natural compactification for the genus 2 maps, which should similarly lead to the proof of the mirror symmetry prediction in genus 2. This work is described in Section 1 below.

The theory of real GW-invariants has long lagged behind the complex theory. These invariants, first defined in genus 0 cases in [W1, W2], should count pseudo-holomorphic maps from symmetric Riemann surfaces commuting with the involutions on the domain and the target. In the recent work of [GZ], the real GW-invariants are defined in all genera for many symplectic manifolds, including all odd-dimensional complex projective spaces and the quintic 3-fold. In [NZ], A. Zinger and I establish a formula that transforms real GW-invariants of many symplectic 3-folds into signed integer counts of smooth real curves. Our formula gives rise to many enumerative results, e.g. in $\mathbb{P}^3$ there are at least 10 genus 2 degree 7 real curves passing through 7 general pairs of conjugate points and at least 40 genus 5 degree 8 real curves passing through 8 general pairs of conjugate points. As a byproduct of our formula, we obtain implications for certain Hodge integrals. This work is described in Section 2 below.

1 Compactification of Genus 2 Moduli Spaces

Let $(X, \omega)$ be a compact symplectic manifold. An almost complex structure on $X$ is a bundle endomorphism $J : TX \to TX$ so that $J^2 = -\text{id}$. A pseudo-holomorphic map $u : \Sigma \to X$ is a smooth map from a (possibly nodal) Riemann surface $\Sigma$ with complex structure $j$ solving the Cauchy-Riemann equation

$$\bar{\partial}_j u = \frac{1}{2} (du + J \circ du \circ j) = 0.$$
For $g, k \in \mathbb{Z}_{\geq 0}$, $B \in H_2(X; \mathbb{Z})$, and an $\omega$-tame almost complex structure $J$, let

$$\overline{\mathcal{M}}_{g,k}(X, B; J) \supset \mathcal{M}_{g,k}^0(X, B; J)$$

be the moduli space of stable genus $g$ pseudo-holomorphic maps with $k$ marked points in the homology class $B$ and the subspace of maps from smooth domains, respectively. While the compact moduli space $\overline{\mathcal{M}}_{g,k}(X, B; J)$ determines a rational homology class (virtual fundamental class) independent of $J$, in general $\overline{\mathcal{M}}_{g,k}(X, B; J)$ can be highly singular and does not contain $\mathcal{M}_{g,k}^0(X, B; J)$ as a dense subset (for any choice of $J$).

The subspace $\mathcal{M}_{g,k}^0(X, B; J)$ is dense in $\overline{\mathcal{M}}_{g,k}(X, B; J)$ if $g = 0$ and $J$ is sufficiently regular, but this is not the case if $g \geq 1$. This raises the question of whether there exists a natural subspace

$$\overline{\mathcal{M}}_{g,k}(X, B; J) \subset \overline{\mathcal{M}}_{g,k}(X, B; J)$$

containing $\mathcal{M}_{g,k}^0(X, B; J)$ such that $\mathcal{M}_{g,k}^0(X, B; J)$ is dense in $\overline{\mathcal{M}}_{g,k}(X, B; J)$ whenever $J$ is sufficiently regular. The naturality condition should include

- for every compact submanifold $(Y, \omega, J)$ of $(X, \omega, J)$
  $$\overline{\mathcal{M}}_{g,k}^0(Y, B; J) = \overline{\mathcal{M}}_{g,k}(X, B; J) \cap \overline{\mathcal{M}}_{g,k}(Y, B; J),$$

- the pre-image of $\overline{\mathcal{M}}_{g,k}^0(X, B; J)$ under the forgetful map
  $$\overline{\mathcal{M}}_{g,k+1}(X, B; J) \rightarrow \overline{\mathcal{M}}_{g,k}(X, B; J)$$

is $\overline{\mathcal{M}}_{g,k+1}^0(X, B; J)$.

For $g = 1$, such a closed subspace is constructed in [Z1].

**Theorem 1 ([N]).** The answer to the above question is affirmative for $g = 2$.

In contrast to its genus 1 counterpart, the boundary strata of $\overline{\mathcal{M}}^0_{2,k}(X, B; J)$ are characterized by numerous distinct conditions, in part because of the various topological types of genus 2 curves and of the existence of Weierstrass points. All boundary strata of $\overline{\mathcal{M}}^0_{2,k}(X, B; J)$ are of the expected virtual dimension. The compactification $\overline{\mathcal{M}}^0_{2,k}(X, B; J)$ of $\mathcal{M}_{0,2}^0(X, B; J)$ is analogous to the compactification by arithmetic genus of [V], but is in arbitrary dimensions and does not require the integrability of $J$.

I am currently formulating the conditions defining $\overline{\mathcal{M}}^0_{2,k}(X, B; J)$ so that the desingularization of $\overline{\mathcal{M}}^0_{2,k}(\mathbb{P}^n, d)$ by sequential blowups along its boundary strata would be feasible. In the genus 1 case, the desingularization $\overline{\mathcal{M}}^0_{1,k}(\mathbb{P}^n, d)$ of $\mathcal{M}_{1,k}^0(\mathbb{P}^n, d)$ with its localization data of [VZ] leads to an effective computation of the GW-invariants of complete intersections. An algebro-geometric approach to $\overline{\mathcal{M}}^0_{1,k}(\mathbb{P}^n, d)$ is provided in [HuL1]. A similar approach to a partial desingularization of $\overline{\mathcal{M}}^0_{2,k}(\mathbb{P}^n, d)$ is introduced in [HuL2], but it does not contain localization data or blowups along certain boundary strata of $\overline{\mathcal{M}}^0_{2,k}(\mathbb{P}^n, d)$. 

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Further research. The construction of $\overline{M}_{2,k}^0(X, B; J)$ is related to the genus 2 case of the prediction of [BCOV] and [HKQ] for the GW-invariants of the quintic 3-fold, whose mathematical proof has remained open so far. The recent work of [CLL] may indicate a possible approach to the GW-invariants of the quintic 3-fold in all genera. A different possible approach is to mimic the idea of the proof [Z3] of the genus 1 BCOV prediction, based on the construction of $\overline{M}_{2,k}^0(X, B; J)$. In pursuit of this goal I intend to

(P1) construct reduced genus 2 GW-invariants arising from $\overline{M}_{2,k}^0(X, B; J, \nu)$ with certain perturbations $\nu$ of the Cauchy-Riemann equation (c.f. [Z2]),

(P2) establish a Quantum Lefschetz Hyperplane property (c.f. [LZ]) for these invariants, and

(P3) obtain a desingularization of the moduli space $\overline{M}_{2,k}^0(\mathbb{P}^n, d)$.

This would lead to a confirmation of the genus 2 mirror symmetry prediction for the GW-invariants of the quintic 3-fold.

2 Real GW-invariants

A compact real symplectic manifold $(X, \omega, \phi)$ consists of a compact symplectic manifold $(X, \omega)$ and an anti-symplectic involution $\phi$ (i.e. $\phi^* \omega = -\omega$). The almost complex structures $J$ are chosen so that $\phi^* J = -J$. A symmetric surface $(\Sigma, \sigma)$ is a (possibly nodal) Riemann surface $\Sigma$ with an orientation-reversing involution $\sigma$. A real pseudo-holomorphic map is a pseudo-holomorphic map $u : \Sigma \to X$ such that $u \circ \sigma = \phi \circ u$; its image in $X$ is a real curve. The main object in the real GW-theory of [GZ] is the moduli space $\overline{M}_{g,l}(X, B; J)^\phi$ of stable real pseudo-holomorphic maps from genus $g$ symmetric surfaces with $l$ conjugate pairs of marked points in the homology class $B \in H_2(X; \mathbb{Z})$. The real moduli space is potentially non-orientable, but for a large family of real symplectic manifolds, including $\mathbb{P}^{2n-1}$ and the quintic 3-fold, a real orientation of [GZ] on $X$ endows $\overline{M}_{g,l}(X, B; J)^\phi$ with an orientation and a virtual class and thus gives rise to real GW-invariants of arbitrary genus for $X$.

In the complex GW-theory, the Fano case of the Gopakumar-Vafa prediction (conjectured in [P], proved in [Z4]) implies that the positive-genus GW-invariants for a Fano class $B$ ($c_1(B) > 0$) of a compact symplectic 3-fold can be canonically expressed in terms of integer counts of curves. In [NZ], A. Zinger and I establish a similar formula that transforms real positive-genus GW-invariants for a Fano class $B$ of a compact real-oriented symplectic 3-fold into signed integer counts of smooth real curves. These integer invariants provide lower bounds for the actual counts of such curves.

Suppose $X$ is a compact real-oriented symplectic 3-fold and $B \in H_2(X; \mathbb{Z})$ is a Fano class. For $g, l \in \mathbb{Z}^{\geq 0}$, let $\overline{M}_{g,l}^*(X, B; J)^\phi \subset \overline{M}_{g,l}(X, B; J)^\phi$ be the subspace consisting of simple (i.e. not multiple-covered) real maps from smooth domains. The constraints $\mu_i \in H^*(X; \mathbb{Z})$, $1 \leq i \leq l$, are taken so that their dimensions sum up to $c_1(B) + 2l$. A tuple $f = (f_i : Y_i \to X)_{i=1}^l$ of pseudocycle representatives for the Poincare duals of $\mu_1, \ldots, \mu_l$ cuts out a space

$\overline{M}_{g,f}^*(X, B; J)^\phi = \{ ([u, (z_i^+, z_i^-)]_{i=1}^l, (y_i)]_{i=1}^l) \in \overline{M}_{g,l}^*(X, B; J)^\phi \times \prod_{i=1}^l Y_i : u(z_i^+) = f_i(y_i) \ \forall \ i \}$.

In [NZ], we show that for a generic choice of $J$ (depending on $g$ and $B$) and each $h \leq g$
• the moduli space $\mathcal{M}_{h,f}(X, B; J)^\phi$ consists of regular maps and
• for a generic choice of $f$, $\mathcal{M}_{h,f}(X, B; J)^\phi$ is a finite set of regular pairs $\{(u, (z_i^+, z_i^-)_{i=1}^l, y_i)_{i=1}^l\}$ such that $u$ is an embedding.

The signed cardinality $\pm|\mathcal{M}_{h,f}(X, B; J)^\phi|$ is proved to be independent of the choice of $J$ and $f$ and can be denoted by $E_{h,B}^{X,\phi}(\mu_1, \ldots, \mu_l)$. This is a signed integer count of genus $h$ real curves.

**Theorem 2** ([NZ]). With notation as above, for each $g \geq 0$ the real GW-invariant satisfies

$$GW_{g,B}^{X,\phi}(\mu_1, \ldots, \mu_l) = \sum_{0 \leq h \leq g \atop g-h \in 2\mathbb{Z}} \tilde{C}_{h,B}(\frac{g-h}{2})E_{h,B}^{X,\phi}(\mu_1, \ldots, \mu_l),$$

where $\tilde{C}_{h,B}(\frac{g-h}{2})$ satisfies the generating function

$$\sum_{g=0}^{\infty} \tilde{C}_{h,B}(g)t^{2g} = \left(\frac{\sinh(t/2)}{t/2}\right)^{h-1+c_1(B)/2}.$$

In [NZ], we compute the real GW-invariants for $\mathbb{P}^3$ with conjugate pairs of point constraints up to $g \leq 5$ and $d \leq 8$ by equivariant localization and transform them into the signed integer counts. These integers provide non-trivial lower bounds for counts of real curves in $\mathbb{P}^3$. The genus 0 numbers coincide with Welschinger’s invariants [W2].

We also obtain some implications for certain Hodge integrals from Theorem 2. For example, let $E$ and $\psi_1$ be the Hodge bundle and the first Chern class of the universal cotangent bundle over $\overline{\mathcal{M}}_{g,1}$, respectively. Then for arbitrary formal variables $x$ and $y$

$$1 + \sum_{g=1}^{\infty} t^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\Lambda(x+y)\Lambda(x)\Lambda(y)}{(x+y)(x+y-\psi_1)} = \frac{\sin(t/2)}{t/2}, \quad \text{where} \quad \Lambda(x) = \sum_{r=0}^{g} (-1)^rc_r(E)x^{g-r}.$$

The specialization $y = -2x$ is equivalent to the $k = -2$ case of [FP, Theorem 2].

**Further research.** A goal of my research is to establish the (genus 1) mirror symmetry for real GW-invariants. Parallel to the approach described in Section 1, it should be possible to define the reduced real GW-invariants and the real version of the hyperplane relation for the quintic 3-fold in the genus 1 case. In light of the localization computation in [PoZ], the proof of the genus 1 mirror symmetry for real GW-invariants of the quintic 3-fold would then be completed.

**References**


