

A proof of Pilgrim's conjecture

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Definition

A *Thurston map* is a pair (f, P_f) where $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is an orientation-preserving branched self-cover of \mathbb{S}^2 of degree $d_f \geq 2$ and P_f is a finite forward invariant set that contains all critical values of f .

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In particular, the branched cover f must be postcritically finite.

Thurston equivalence

Definition

Two Thurston maps f and g are combinatorially equivalent if and only if there exist two homeomorphisms $h_1, h_2: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that the diagram

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{S}^2, P_g) \\ \downarrow f & & \downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_2} & (\mathbb{S}^2, P_g) \end{array}$$

commutes, $h_1|_{P_f} = h_2|_{P_f}$, and h_1 and h_2 are homotopic relative to P_f .

Thurston's theorem

Theorem (Thurston's Theorem)

A postcritically finite branched cover $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ with hyperbolic orbifold is either Thurston-equivalent to a rational map g (which is then necessarily unique up to conjugation by a Möbius transformation), or f has a Thurston obstruction.

Some further notations

- \mathcal{T}_f is the Teichmüller space modeled on the marked surface (\mathbb{S}^2, P_f)
- \mathcal{M}_f is the corresponding moduli space
- Recall that \mathcal{T}_f can be defined as the quotient of the space of all diffeomorphisms from (\mathbb{S}^2, P_f) to the Riemann sphere. We write $\tau = \langle h \rangle$ if a point τ is represented by a homeomorphism h
- $Q(\mathbb{P}, h(P_f))$ is the cotangent space at a point $\tau = \langle h \rangle$ in the Teichmüller space \mathcal{T}_f which is canonically isomorphic to the space of all integrable meromorphic quadratic differentials on the marked Riemann surface corresponding to τ

Maps between Teichmüller spaces

Pullback map

Suppose we have a (unbranched) covering map $h: A \rightarrow B$ between finite type surfaces A and B . Then we can define $h^*: \mathcal{T}(B) \rightarrow \mathcal{T}(A)$ that acts by pulling back complex structures from B to A .

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Projection map

Suppose we have an inclusion map $i: A \rightarrow B$ between finite type surfaces A and B . This happens exactly when A can be obtained from B by removing finitely many points. Then we can define the forgetful projection $i_*: \mathcal{T}(A) \rightarrow \mathcal{T}(B)$ which “forgets” the positions of extra punctures.

Thurston's iteration

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In our setting we have the unbranched cover

$f: \mathbb{S}^2 \setminus f^{-1}(P_f) \rightarrow \mathbb{S}^2 \setminus P_f$ and the identity injection

$\text{id}: \mathbb{S}^2 \setminus f^{-1}(P_f) \rightarrow \mathbb{S}^2 \setminus P_f$ since $f^{-1}(P_f) \supset P_f$. Denote

$\sigma_f = \text{id}_* \circ f^*: \mathcal{T}_f \rightarrow \mathcal{T}_f$.

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Another definition of Thurston's iteration

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{P}, h_1(P_f)) \\ \downarrow f & & \downarrow f_\tau \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_\tau} & (\mathbb{P}, h_\tau(P_f)) \end{array} \quad (1)$$

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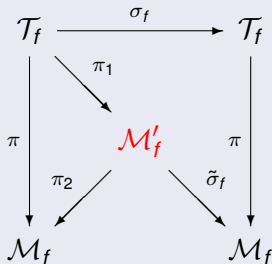
Lemma

A Thurston map f is equivalent to a rational function if and only if σ_f has a fixed point.

Lemma (Douady, Hubbard)

There exists an intermediate cover \mathcal{M}'_f of \mathcal{M}_f (so that $\mathcal{T}_f \xrightarrow{\pi_1} \mathcal{M}'_f \xrightarrow{\pi_2} \mathcal{M}_f$ are covers and $\pi_2 \circ \pi_1 = \pi$) such that

- i. π_2 is finite,
- ii.



commutes for some map $\tilde{\sigma}_f: \mathcal{M}'_f \rightarrow \mathcal{M}_f$,

- iii. If $\pi_1(\tau_1) = \pi_1(\tau_2)$ then $f_{\tau_1} = f_{\tau_2}$ up to pre- and post-composition by Moebius transformations.

Definition

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Lemma

σ_f is a holomorphic self-map of \mathcal{T}_f and the co-derivative of σ_f satisfies $(d\sigma_f(\tau))^* = (f_\tau)_*$ where $(f_\tau)_*$ is the push-forward operator on quadratic differentials.

Metric definitions

For a meromorphic integrable quadratic differential on \mathbb{P} we define

- its Teichmüller norm

$$\|q\|_{\mathcal{T}} = 2 \int_{\mathbb{P}} |q|$$

and

- its Weil-Petersson norm

$$\|q\|_{WP} = \left(\int_{\mathbb{P}} \rho^{-2} |q|^2 \right)^{1/2}$$

Estimates on the norm of $d\sigma_f^*$

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$$\int_U |g_* q| = \int_U \left| \sum_i g_i^* q \right| \leq \sum_i \int_U |g_i^* q| = \sum_i \int_{U_i} |q|$$



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Corollary (almost)

There exists at most one fixed point of σ_f , hence the uniqueness in Thurston's theorem follows.

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Corollary

σ_f is Lipschitz with respect to the WP-metric.

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- A multicurve Γ is *completely invariant* if $f^{-1}(\Gamma) = \Gamma$

Structure of the boundary of the augmented Teichmüller space

The augmented Teichmüller space $\overline{\mathcal{T}}_f$ - the set of all marked noded stable Riemann surfaces of the same type

- The augmented Teichmüller space $\overline{\mathcal{T}}_f$ is a stratified space with strata \mathcal{S}_Γ corresponding to different multicurves Γ on (\mathbb{S}^2, P_f) . In particular, $\mathcal{T}_f = \mathcal{S}_\emptyset$.

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- Each point in the stratum \mathcal{S}_Γ is a collection of complex structures on the components of the corresponding topological noded surface with marked points.
- \mathcal{S}_Γ is the product of Teichmüller spaces of these components.
- Within each stratum one can define its own natural Teichmüller and Weil-Petersson metrics.

Augmented Teichmüller space

Lemma

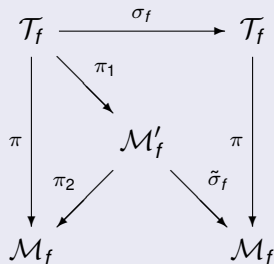
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Recall the diagram

$$\begin{array}{ccc} \overline{\mathcal{T}}_f & \xrightarrow{\sigma_f} & \overline{\mathcal{T}}_f \\ \pi \downarrow & \searrow \pi_1 & \downarrow \pi \\ & \overline{\mathcal{M}}'_f & \\ \pi_2 \swarrow & & \searrow \tilde{\sigma}_f \\ \overline{\mathcal{M}}_f & & \overline{\mathcal{M}}_f \end{array}$$

Theorem (Masur)

The augmented Teichmüller space $\overline{\mathcal{T}}_f$ is homeomorphic to the WP-completion of the Teichmüller space.

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Corollary

σ_f extends continuously to $\overline{\mathcal{T}}_f$.

Definition of σ_f on the boundary

We represent points in $\overline{\mathcal{T}}_f$ not only by homeomorphisms but also by continuous maps from (\mathbb{S}^2, P_f) to a noded Riemann surface that are allowed to send a whole simple closed curve to a node. Consider such an h representing some point in $\overline{\mathcal{T}}_f$.

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We complete this diagram as before

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (R_1, P_f) \\ \downarrow f & & \downarrow \{f^{C_i}\} \\ (\mathbb{S}^2, P_f) & \xrightarrow{h} & (R, P_f) \end{array}$$

Action of σ_f on $\overline{\mathcal{T}}_f$

Theorem

The map σ_f as defined above is continuous on $\overline{\mathcal{T}}_f$.

Remark.

Note that by definition σ_f maps any stratum \mathcal{S}_Γ into the stratum $\mathcal{S}_{f^{-1}(\Gamma)}$, therefore invariant boundary strata are in one-to-one correspondence with completely invariant multicurves.

Thurston matrix and obstructions

Definition

Denote by \mathcal{C} the set of all homotopy classes of essential simple closed curve. Define Thurston linear operator $M: \mathbb{R}^{\mathcal{C}} \rightarrow \mathbb{R}^{\mathcal{C}}$ by setting

$$M(\gamma) = \sum_{f(\gamma_i)=\gamma} \frac{1}{\deg f|_{\gamma_i}} \gamma_i.$$

Every multicurve Γ has its associated *Thurston matrix* M_{Γ} which is the restriction of M to \mathbb{R}^{Γ} .

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Definition

Since all entries of M_{Γ} are non-negative real, the leading eigenvalue λ_{Γ} of M_{Γ} is also real and non-negative. A multicurve Γ is a *Thurston obstruction* if $\lambda_{\Gamma} \geq 1$.

Definition

We call Γ *simple* if there exists a leading eigenvector of M_Γ with positive coordinates. Each multicurve has a simple sub-multicurve with the same leading eigenvalue.

Dynamics near boundary strata

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Definition

An invariant stratum \mathcal{S} of $\overline{\mathcal{T}}_f$ is *weakly attracting* if there exists a nested decreasing sequence of neighborhoods U_n such that $\sigma_f(U_n) \subset U_n$ and $\bigcap U_n = \mathcal{S}$.

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Definition

An invariant stratum \mathcal{S} of $\overline{\mathcal{T}}_f$ is *weakly repelling* if for any compact set $K \subset \mathcal{S}$ there exists a neighborhood $K \subset U$ such that every point of $U \cap \mathcal{T}_f$ escapes from U after finitely many iterations.

Classification of invariant boundary strata

Lemma

If $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_m\}$ is a completely invariant simple multicurve and $\lambda_\Gamma \geq 1$, then S_Γ is weakly attracting. Otherwise it is weakly repelling.

Sketch of the proof of Thurston's theorem

Pick any starting point $\tau \in \mathcal{T}_f$ and consider $\tau_n = \sigma_f^n(\tau)$. Take an accumulation point in $\overline{\mathcal{M}}_f$ of the projection of τ_n to the moduli space on the stratum of smallest possible dimension.

For simplicity we assume that τ_n accumulates on some $\tau_0 \in \mathcal{S}_\Gamma$.

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- If $\Gamma = \emptyset$ then τ_0 is a fixed point of σ_f .
- If $\Gamma \neq \emptyset$ then Γ must be a Thurston obstruction. Otherwise, \mathcal{S}_Γ is weakly repelling and therefore τ_n can not have an accumulation point there.

Definition

The *canonical* obstruction Γ_f is the set of all homotopy classes of curves γ that satisfy $l(\gamma, \sigma_f^n(\tau)) \rightarrow 0$ for all $\tau \in \mathcal{T}_f$

Pilgrim's theorems

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Theorem (Canonical Obstruction Theorem)

If for a Thurston map with hyperbolic orbifold its canonical obstruction is empty then it is Thurston equivalent to a rational function. If the canonical obstruction is not empty then it is a Thurston obstruction.

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For any point $\tau \in \mathcal{T}_f$ there exists a bound $L = L(\tau, f) > 0$ such that for any essential simple closed curve $\gamma \notin \Gamma_f$ the inequality $l(\gamma, \sigma_f^n(\tau)) \geq L$ holds for all n .

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Pilgrim's conjecture

Recall that the action on any invariant stratum is given by pullbacks of complex structures by a collection of maps σ_{fC} for all components C of any surface in the stratum. Combinatorics of the process is very simple: we have a map from a finite set into itself, every component is pre-periodic. The whole action, therefore, can be characterized by studying cycles of components. For each cycle Y there are three cases, the composition f^Y of all coverings in the cycle is either of the following:

- a homeomorphism,
- a Thurston map with a parabolic orbifold,
- a Thurston map with a hyperbolic orbifold.

Theorem

If a cycle Y of components a topological surface corresponding to the stratum \mathcal{S}_Γ , has hyperbolic orbifold then f^Y is not obstructed and, hence, equivalent to a rational map.

Pilgrim's conjecture

Theorem

If a cycle Y of components a topological surface corresponding to the stratum \mathcal{S}_{Γ_f} has hyperbolic orbifold then f^Y is not obstructed and, hence, equivalent to a rational map.

Idea of the proof

We may assume that our component is mapped to itself. Take an accumulation point $\tau_0 \in \mathcal{S}_{\Gamma_f}$ of $\sigma_f^n(\tau)$. Let τ' be the coordinate corresponding to Y . Let Γ_Y be the canonical obstruction for f^Y .

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Then the accumulation set of $\sigma_f^n(\tau')$ must be a subset of the closure of $\mathcal{S}_{\Gamma_f \cup \Gamma_Y}$. On the other hand, it is clearly a subset of the accumulation set of $\sigma_f^n(\tau) \subset \mathcal{S}_{\Gamma_f}$. This means Γ_Y must be empty.

Generalized Pilgrim's conjecture

Theorem

If a cycle Y of components a topological surface corresponding to the stratum \mathcal{S}_{Γ_f} is a Thurston map then the canonical obstruction of f^Y is empty.

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This, with some extra effort, leads to a complete topological description of canonical obstructions.