



On Thurston's characterization theorem for branched covers

by

Nikita Selinger

A Thesis submitted in partial fulfillment
of the requirements for the degree of

**Doctor of Philosophy
in Mathematics**

Approved, Thesis Committee

Prof. Dr. Dierk Schleicher

Name and title of chair

Prof. Dr. John Hubbard

Name and title of committee member

Prof. Dr. Daniel Meyer

Name and title of committee member

Prof. Dr. Étienne Ghys

Name and title of committee member

Date of Defense: July 7, 2011

School of Engineering and Science

Abstract

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Nikita Selinger

Doctor of Philosophy in Mathematics

Jacobs University Bremen

Professor Dierk Schleicher, Chair

Let f be a postcritically finite branched self-cover of a 2-dimensional topological sphere. Such a map induces an analytic self-map σ_f of a finite-dimensional Teichmüller space. We prove that this map extends continuously to the augmented Teichmüller space and give an explicit construction for this extension. This allows us to characterize the dynamics of Thurston's pullback map near invariant strata of the boundary of the augmented Teichmüller space. The resulting classification of invariant boundary strata is used to prove a conjecture by Pilgrim and to infer further properties of Thurston's pullback map. We obtain a complete topological description of canonical obstructions. Our approach also yields new proofs of Thurston's theorem and Pilgrim's Canonical Obstruction theorem.

Acknowledgments

I thank my advisor, Dierk Schleicher, for his constant support and guidance.

I thank all people who also helped me with learning mathematics and doing research, including Dima Dudko, Adam Epstein, Alexandre Eremenko, Ursula Hamenstädt, John Hubbard, Sarah Koch, Misha Lyubich, Kevin Pilgrim, Mitsuhiro Shishikura, Vladlen Timorin, Misha Yampolsky.

I thank Deutsche Forschungsgemeinschaft, Jacobs University Bremen, CODY network, the German-Israeli foundation, and the Fields Institute in Toronto for financial support during my studies.

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Chapter 1

Introduction

In the early 1980's Thurston proved one of the most important theorems in the field of Complex Dynamics. His characterization theorem provides a topological criterion of whether a given combinatorics can be realized by a rational map, and also provides a rigidity statement: two rational maps are equivalent if and only if they are conjugate by a Moebius transformation.

The original proof from [DH93] of Thurston's characterization theorem (Theorem 3.1.2) relates the original question to whether or not Thurston's pullback map on a Teichmüller space has a fixed point. In [DH93] the authors study the behavior of Thurston's pullback map near infinity without specifying any structure at the boundary of the considered Teichmüller space. The question of finding the notion of the boundary for the Teichmüller space that would be appropriate for the problem is very natural, and was in the air since the first proof of the theorem came out.

One obvious candidate to consider is the Thurston boundary which has been successfully used by Thurston to give a similar but considerably simpler proof of the characterization theorem for surface diffeomorphisms. The analytic self-map of the Teichmüller space investigated in this case extends continuously to the Thurston boundary of the Teichmüller space. This yields a continuous self-map of a topological ball. The proof then uses Brouwer's fixed point theorem as an essential ingredient.

We feel that the following gives some indication of why the characterization of rational maps is a more complicated matter than the characterization of diffeomorphisms.

Theorem 1.0.1. *There exist postcritically finite branched covers f such that Thurston's pullback map does not extend to the Thurston boundary of the Teichmüller space.*

In the case when the Teichmüller space is one dimensional, Thurston's pullback map can be described explicitly as a self-map of the unit disk \mathbb{D} . One immediately notices that for the map $f(z) = 3z^2/(2z^3 + 1)$ considered in [BEKP09], Thurston's pullback map can not be continued to a self-map of $\overline{\mathbb{D}}$ which is naturally homeomorphic to the Thurston compactification of the Teichmüller space. Note that in some cases, such an extension is possible. For instance, in the same article [BEKP09] the authors present an example by McMullen of branched covers with constant Thurston's pullback map; other examples are Lattès maps for which the pullback maps are automorphisms of \mathbb{D} . In Section 4.4 we give a more conceptual proof of Theorem 4.4.4.

From our point of view, Theorem 4.4.4 just says that the Thurston boundary is not the right boundary notion for the task. The next theorem has significantly more consequences and is in the heart of the whole thesis.

Theorem 1.0.2. *Thurston's pullback map extends continuously to a self-map of the augmented Teichmüller space.*

The topology of the augmented Teichmüller space is by far more complicated than the topology of the compactification of the Teichmüller space with the Thurston boundary. It is not compact or even locally compact, so that we can not apply tools like Brouwer's fixed point theorem. It is not true that Thurston's pullback map must always have a fixed point in the augmented Teichmüller space (the simplest counterexample is a Lattès map corresponding to a matrix with distinct real eigenvalues, see [DH93] for a definition), however, this is true in many cases (see Theorem 4.5.3).

In Section 4.1, we define the extension to the boundary of the augmented Teichmüller space explicitly in a way that is similar to the definition of the action of Thurston's pullback map on the Teichmüller space. This brings, in our opinion, new insights in understanding the behavior of Thurston's pullback map. In Section 4.2 we characterize the dynamics of Thurston's pullback map near invariant strata on the boundary of the augmented Teichmüller space. In Section 4.3 we use the obtained classification to simplify the proofs of Thurston's theorem and Canonical Obstruction theorem due to Pilgrim (see Theorem 3.2.3).

In Section 4.5, an application of our approach is given: we prove a conjecture by Kevin Pilgrim [Pil03] (see Theorem 4.5.2). In Section 4.6, we give a complete topological description of canonical obstructions.

Chapter 2

Preliminaries

2.1 Extremal length of a family of curves

Let Γ be a family of Jordan curves in the complex plane. We define a conformal invariant for this family – its extremal length $\lambda(\Gamma)$.

Consider on \mathbb{C} an arbitrary measurable conformal metric $\rho|dz|$ with finite total mass

$$m_\rho = \int_{\mathbb{C}} \rho^2 |dz|^2$$

(we will call such metrics admissible). Denote

$$l_\rho(\gamma) = \int_\gamma \rho |dz|$$

the length of $\gamma \in \Gamma$ in this metric (the length is infinite if γ is non-rectifiable or ρ is non-integrable over γ). Let

$$l_\rho(\Gamma) = \inf_{\gamma \in \Gamma} l_\rho(\gamma).$$

We can normalize this quantity so that it becomes invariant under the change of scale:

$$\lambda_\rho(\Gamma) = \frac{l_\rho^2(\Gamma)}{m_\rho}.$$

Definition 2.1.1. The *extremal length* of a family Γ is the following quantity:

$$\lambda(\Gamma) = \sup_{\rho} \lambda_\rho(\Gamma),$$

where supremum is taken over all admissible metrics.

2.2 Quasiconformal mappings

Quasiconformal mappings are one of the basic objects in Teichmüller theory. By definition, a conformal mapping is one that preserves infinitesimal circles. For our purposes, we have to extend this class of maps.

Suppose at first, that we have an orientation-preserving (real) diffeomorphism $f(z)$, mapping the complex plane to itself. Let us write its differential in the following form:

$$df(z) = pdz + qd\bar{z},$$

where p, q are formal derivatives:

$$p = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), q = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

In this case infinitesimal circles are mapped to infinitesimal ellipses. We can easily estimate the ratio of the axes of the image. The inequality:

$$||p| - |q|| |dz| \leq |df| \leq (|p| + |q|) |dz|$$

shows that the ratio of the axes is

$$K = \frac{|p| + |q|}{||p| - |q||}.$$

This ratio computed at a specific point is also called a *dilatation*.

Definition 2.2.1. An orientation-preserving diffeomorphism is *quasiconformal* if its dilatation is uniformly bounded.

The Jacobian of f is equal to:

$$J(f) = |p|^2 - |q|^2.$$

As we consider only orientation-preserving diffeomorphisms, the Jacobian is always positive. The maximal dilatation K of a quasiconformal map f is defined as:

$$K(f) = \sup \frac{|p| + |q|}{|p| - |q|}.$$

Quite often it is more convenient to work with the maximal eccentricity $k(f)$ of f

which is introduced by equality:

$$k(f) = \frac{K-1}{K+1} = \sup \frac{|q|}{|p|}.$$

It follows from definitions that $1 \leq K < \infty$ and $0 \leq k < 1$. The mapping is conformal if and only if $K = 1, k = 0$. The diffeomorphism f is K -quasiconformal if $K(f) \leq K$.

Though the definition above broadens substantially the class of conformal mappings, it is not flexible enough. The differentiability requirement is too strong. In order to be able to introduce the powerful machinery of Teichmüller theory, we still need a wider class of mappings. There are both geometric and analytic approaches in defining general quasiconformal maps. The main idea is to replace differentiability with a softer condition.

A few analytical definitions can be found in [H]. We cite verbatim the very first one:

Definition 2.2.2. Let U, V be open subsets of \mathbb{C} , take $K \geq 1$, and set $k = (K - 1)/(K + 1)$, so that $0 \leq k < 1$. A mapping $f : U \rightarrow V$ is K -quasiconformal if it is a homeomorphism whose distributional partial derivatives are in L^2_{loc} (locally in L^2) and satisfy

$$\left| \frac{\partial f}{\partial \bar{z}} \right| \leq k \left| \frac{\partial f}{\partial z} \right|$$

in L^2_{loc} , i.e., almost everywhere.

A map is *quasiconformal* if it is K -quasiconformal for some K .

In order to illustrate the geometric meaning of quasiconformality we also give a geometric definition.

Definition 2.2.3. A *quadrilateral* is a closed Jordan disk together with a marked pair of opposite sides on its boundary.

Suppose a quadrilateral is conformally equivalent to a rectangle R with sides a and b so that the marked sides are mapped on the sides of length b . Then one can define the modulus of this quadrilateral to be equal to the ratio $m = a/b$. There is the following equivalent definition:

Definition 2.2.4. The *modulus* $\text{mod}(Q)$ of a quadrilateral Q is the extremal length of the family of curves in the quadrilateral connecting its marked sides.

Then the following theorem holds:

Theorem 2.2.1 (Geometric definition of quasiconformal mappings). *A homeomorphism f is K -quasiconformal if and only if for any quadrilateral Q the following inequality holds:*

$$\frac{1}{K} \operatorname{mod}(Q) \leq \operatorname{mod} f(Q) \leq K \operatorname{mod}(Q).$$

2.3 Conformal (complex) structures and Beltrami differentials

Another notion of Teichmüller theory, which is intimately connected to quasiconformal mappings, is the notion of Beltrami differentials. Recall that a differential of type (m, n) can be viewed as a rule which, for any local parameter z defined in a domain $D \subset S$, provides a measurable function $\psi(z)$ in such a way that the expression $\psi(z) dz^m d\bar{z}^n$ is invariant under a conformal change of coordinates. The absolute value $|\mu(z)|$ of a differential $\mu(z) d\bar{z}/dz$ of type $(-1, 1)$ is a scalar function on S . In the case $|\mu| \leq k < 1$ we will call $\mu(z)$ a Beltrami coefficient and $\mu d\bar{z}/dz$ - a Beltrami differential.

By a measurable conformal structure on a domain $U \subset \mathbb{C}$ we mean a measurable family of conformal structures in the tangent planes $T_z U, z \in U$. In other words, it is a measurable family of infinitesimal ellipses $E(z) \in T_z U$, defined up to scaling by a positive measurable function. (In all cases where we talk about measurable objects, they are defined almost everywhere.) Since $T_z U$ is isomorphic to \mathbb{C} , the ratio K of major and minor axes of the infinitesimal ellipse is uniquely defined in (almost) every point of U . Define a measurable function $\mu : U \rightarrow \mathbb{D}$ as follows: $\mu(z) = \frac{K-1}{K+1}$. Thus, for every conformal structure there exists a corresponding Beltrami coefficient $\mu(z), z \in U$. That means that conformal structures on U can be analytically described as elements μ of the unit ball of the space $L^\infty(U)$. The standard complex structure σ is represented by a family of infinitesimal circles; the corresponding Beltrami coefficient is identically equal to zero.

For an orientation-preserving diffeomorphism we define the corresponding Beltrami coefficient using the formula: $\mu_f = \frac{df_{\bar{z}}}{df_z}$. The same formula, however, is also applicable in the case of an arbitrary quasiconformal map since quasiconformal maps are differentiable almost everywhere which is sufficient for defining the Beltrami coefficient.

On the other hand, suppose we are given a Beltrami coefficient μ . Define the corresponding quasiconformal mapping to be a solution of the differential equation:

$$\frac{df_\mu}{d\bar{z}} = \mu \frac{df_\mu}{dz}.$$

The following theorem [AB60] plays fundamental role in the study of quasiconformal mappings.

Theorem 2.3.1 (Measurable Riemann mapping theorem). *Let μ be a measurable Beltrami differential on \mathbb{P} . Then there exists a quasiconformal mapping that satisfies the equation $\frac{df_\mu}{d\bar{z}} = \mu \frac{df_\mu}{dz}$. This mapping is unique up to a post-composition with a Möbius transformation. In particular, there exists a unique solution which fixes three selected points in \mathbb{P} (for example, 0, 1 and ∞).*

Since Beltrami differentials are defined in a coordinate-independent way, all the aforementioned can be generalized to arbitrary Riemann surfaces. Thus, for a Riemann surface S there exists a bijection between Beltrami differentials and quasiconformal homeomorphisms on S . Every quasiconformal homeomorphism f defines on S a new conformal structure with Beltrami coefficient μ_f .

2.4 Annuli in Riemann surfaces

Definition 2.4.1. By an *annulus* we mean a doubly connected domain in \mathbb{C} .

The boundary of an annulus consists of two connected components. As is done for quadrilaterals, one can define the modulus of an annulus.

Definition 2.4.2. The *modulus* $\text{mod}(A)$ of an annulus A is the extremal length of the family of all curves connecting the two components of its boundary.

Every annulus is conformally isomorphic to a flat cylinder. If the modulus M of the annulus is finite, then the annulus is isomorphic to the straight cylinder $S^1 \times (0, 2\pi M)$. If the modulus is infinite, there are two options : the annulus is isomorphic either to the infinite cylinder $S^1 \times (0, +\infty)$ or to the bi-infinite cylinder $S^1 \times (-\infty, +\infty)$. The last case is the only possibility of an annulus to be a euclidean surface. All the other annuli are hyperbolic Riemann surfaces. Since extremal length is a conformal invariant, two annuli with different moduli are not conformally isomorphic.

Another possible interpretation represents annuli as domains in \mathbb{C} bounded by two circles $|z| = 1$ and $|z| = e^{-2\pi M}$ (in the case of an infinite modulus we get two standard annuli: the punctured disk and the punctured plane).

The Grötzsch inequality has proved to be very useful in holomorphic dynamics.

Theorem 2.4.1 (The Grötzsch inequality). *If an annulus A contains disjoint annuli A_i , $i = \overline{1, k}$ with $k \in \mathbb{N}$, which are all homotopic to A , then*

$$\text{mod}(A) \geq \sum_i \text{mod}(A_i).$$

The equality holds here if and only if all A_i are round sub-annuli of A and the closure of the union of A_i contains A .

Note that the statement of the previous theorem is not trivial when $k = 1$. We get the following.

Corollary 2.4.2. *If A is a sub-annulus of B , which is homotopic to B , then $\text{mod} A \leq \text{mod} B$ and the equality holds only when $A = B$.*

Definition 2.4.3. The *core curve* of an annulus of finite modulus $2M$ is the curve that separates it into two sub-annuli with moduli M .

By Theorem 2.4.1, such a curve is unique. If A is the straight cylinder $S^1 \times (0, 2\pi M)$, then the core curve of A is the circle $S^1 \times \{\pi M\}$. If A is a round annulus bounded by two circles $|z| = 1$ and $|z| = e^{-2\pi M}$ in \mathbb{C} , then the core curve of A is the circle $|z| = e^{-\pi M}$.

The modulus of an annulus has two important properties:

1. If an annulus A' is the image of an annulus A under the action of a K -quasiconformal mapping, then the following inequality holds:

$$\frac{1}{K} \text{mod}(A) \leq \text{mod}(A') \leq K \text{mod}(A').$$

2. If an annulus A' is the image of an annulus A under the action of a holomorphic unbranched covering of degree d , then the following equality holds:

$$\text{mod}(A') = d \text{mod}(A).$$

We generalize our definition of annuli to include doubly connected domains on any Riemann surface R . The most interesting case is when R is hyperbolic. The hyperbolic length $l(\gamma, R)$ of a simple closed geodesic γ on a hyperbolic Riemann surface R is closely related to the maximal modulus $M(\gamma, R)$ of an annulus in R homotopic to γ . If $A \subset R$ is an annulus homotopic to γ , then the core curve of A is homotopic to γ and, hence, its length is at most $l(\gamma, R)$. On the other hand, since the inclusion map $\text{id}: A \rightarrow R$ is holomorphic, the hyperbolic length of the core

curve in R is at most the hyperbolic length of the core curve in A , which is equal to $\pi/\text{mod } A$. Thus, we see that $M(\gamma, R) \leq \pi/l(\gamma, R)$.

The estimate from below is provided by the following classical theorem (see for example [DH93, Hub06]). We say that γ admits a *collar* of width w if w -neighborhood of γ is an annulus. Define the collar function $\eta(l)$ by the following formula:

$$\eta(l) = \ln \frac{e^{l/2} + 1}{e^{l/2} - 1}.$$

Theorem 2.4.3 (The Collaring Lemma). *Let $\gamma_1, \gamma_2, \dots$ be a collection of disjoint simple closed geodesic on a hyperbolic surface R , each γ_i of length l_i . Then γ_i admits a collar of width $\eta(l_i)$ and these collars are disjoint.*

One can show (see [DH93]) that the modulus of $2\eta(l)$ -collar around a geodesic of length l is at least $\pi/l - 1$. This yields the estimate $M(\gamma, R) \geq \pi/l(\gamma, R) - 1$.

The Collaring Lemma implies that two distinct simple short geodesics can not intersect. Indeed, if some geodesic intersects γ of length l , it must traverse the collar of γ of width $2\eta(l)$. It is straightforward to check that $\eta(l)$ is decreasing and $l = \ln(3 + 2\sqrt{2}) = 2\ln(1 + \sqrt{2})$ is the only solution to $l = 2\eta(l)$.

Corollary 2.4.4. *Any two different simple closed geodesics on a hyperbolic surface R of lengths less than $\ln(3 + 2\sqrt{2})$ are disjoint.*

We can reformulate the previous corollary using the relation between lengths of curves and moduli of annuli around them.

Corollary 2.4.5. *Any two different simple closed geodesics γ_1 and γ_2 on a hyperbolic surface R with $M(\gamma_1, R) > \pi/(\ln(3 + 2\sqrt{2}))$ and $M(\gamma_2, R) > \pi/(\ln(3 + 2\sqrt{2}))$ are disjoint.*

Lemma 2.4.6 (Continuity of modulus). *Let A_i be an annulus in \mathbb{P} with two complementary components B_i and C_i , for every $i \in \mathbb{N}$. Suppose that $B_i \rightarrow B$ and $C_i \rightarrow C$ as $i \rightarrow \infty$ with respect to the Hausdorff metric, and both B and C contain at least two points. If there exists a doubly connected component A of $\mathbb{P} \setminus \{B \cup C\}$ then $\text{mod } A_i \rightarrow \text{mod } A$; otherwise $\text{mod } A_i \rightarrow 0$.*

Proof. First we note that B_i and C_i are connected for every i , therefore B and C are connected. This immediately implies that there exists at most one doubly connected component A in the complement of B and C .

Suppose A exists. We may choose a compactly contained in A and homotopic to A sub-annulus A' such that $\text{mod } A - \text{mod } A' = \varepsilon$ where $\varepsilon > 0$ is arbitrarily

small. Then the two complementary components B' and C' of A' are compact neighborhoods of B and C respectively. For i large enough, we have that $B_i \subset B'$ and $C_i \subset C'$ and, hence, $A_i \supset A'$. We see that $\text{mod } A_i > \text{mod } A' = \text{mod } A - \varepsilon$. We conclude that $\liminf(\text{mod } A_i) \geq \text{mod } A$.

Let $b_1, b_2 \in B$ and $c \in C$ be three distinct points in \mathbb{P} . For i large enough, there exist three distinct points b_1^i, b_2^i and c^i such that $b_1^i, b_2^i \in B_i$ and $c^i \in C_i$, and $d(b_1, b_1^i) < \varepsilon$, $d(b_2, b_2^i) < \varepsilon$ and $d(c, c^i) < \varepsilon$. Letting ε go to 0, we construct a sequence $\{m_i\}$ of Moebius transformations such that $b_1, b_2 \in m_i(B_i)$, $c \in m_i(C_i)$ and $\{m_i\}$ converges uniformly to the identity map on \mathbb{P} . Then the Hausdorff distance between B_i and $m_i(B_i)$ tends to 0 as $i \rightarrow \infty$, and $m_i(B_i) \rightarrow B$ and, analogously, $m_i(C_i) \rightarrow C$. Since $\text{mod } m_i(A_i) = \text{mod } A_i$, it is enough to prove the statement of the lemma for the sequence $\{m_i(A_i)\}$.

Thus, we may assume that $b_1, b_2 \in B_i$ and $c \in C_i$ for all i . Let $a = \limsup(\text{mod } A_i) > 0$; pick a sequence n_i such that $\text{mod } A_{n_i} > a - \varepsilon$ for all $i \in \mathbb{N}$. Consider a sequence of sub-annuli A'_{n_i} of A_{n_i} with $\text{mod } A'_{n_i} = a - \varepsilon$ for all $i \in \mathbb{N}$; let $\{f_i : R \rightarrow A'_{n_i}\}$ be a sequence of conformal isomorphisms from a round annulus R of modulus $a - \varepsilon$ onto A'_{n_i} . Since all f_i do not assume values b_1, b_2 or c on R , the family $\{f_i\}$ is normal by Montel's theorem and there exists a subsequence that converges locally uniformly to a holomorphic map f defined on R . Clearly the diameters of sets B_i tend to the diameter of B and the diameters of sets C_i tend to the diameter of C . The core curve of R is mapped by f_i to a smooth Jordan curve separating B_i and C_i and therefore the lower limit of the diameter of this curve is positive. This yields that f cannot be constant and, hence, is a conformal map onto an annulus that separates b_1, b_2 and c . Let us prove that $f(R) \cap B = \emptyset$.

On contrary, let $f(z) = b \in B$ for some point $z \in R$. To simplify the notation, we assume that $b \neq \infty$ and that $f_i \rightarrow f$. Let r be the distance between z and the boundary of the annulus R . For any $\varepsilon > 0$, if i is large enough there exists a point $b'_i \in B_i$ such that $|b'_i - b| < \varepsilon$ and $|f_i(z) - b| < \varepsilon$. It follows that $|f_i(z) - b'_i| < 2\varepsilon$ but b'_i is not in the image of f_i . Koebe 1/4 theorem implies that $f'_i(z) < 8\varepsilon/r$. Hence, $f'(z) = \lim f'_i(z) = 0$, which contradicts the fact that f is conformal. Therefore $f(R)$ does not intersect B nor, by the same argument, C . We conclude that the component A of the complement of $B \cup C$ containing $f(R)$ is an annulus and $\text{mod } A \geq \text{mod } f(R) = \text{mod } R = a - \varepsilon$. We see that $\limsup(\text{mod } A_i) \leq \text{mod } A$. \square

2.5 Teichmüller and moduli spaces

One of the simplest definition of Teichmüller is given in [Ahl06]. The points in Teichmüller space are equivalence classes of Riemann surfaces. Let S_0 be the base Riemann surface. Consider the set of pairs (S, f) where S is a Riemann surface and $f : S_0 \rightarrow S$ is an orientation preserving quasiconformal homeomorphism. We say that two pairs of this set are equivalent if the corresponding Riemann surfaces are conformally isomorphic and the composition $f_2 \circ f_1^{-1}$ is homotopic to the conformal isomorphism between them. The classes of pairs modulo this relation constitute the Teichmüller space $\mathcal{T}(S_0)$. In particular, with this definition there is an initial point of the space $\mathcal{T}(S_0)$, namely (S_0, I) , where I is the identity map. Note that there are ways to define Teichmüller space so that it has no preferred initial point (see [Hub06]).

Let $\text{QC}(S)$ be the group of orientation-preserving self-homeomorphisms of S and $\text{QC}^0(S)$ be the subgroup of $\text{QC}(S)$ which consists of all homeomorphisms isotopic to the identity map on S . One can check that $\text{QC}^0(S)$ is in fact a subgroup and is normal.

Definition 2.5.1. The *mapping class group* $MCG(S)$ is the following quotient:

$$MCG(X) = \text{QC}(S)/\text{QC}^0(S).$$

The mapping class group $MCG(S_0)$ naturally acts on $\mathcal{T}(S_0)$. Indeed, if g is a homeomorphism representing an element in $MCG(S_0)$ then it sends a pair (S, f) to a pair $(S, f \circ g)$. The *moduli space* $\mathcal{M}(S_0)$ is defined as a quotient of the Teichmüller space $\mathcal{T}(S_0)$ under the action of $MCG(S_0)$.

In our research, we deal with a specific kind of Teichmüller spaces: the Teichmüller space \mathcal{T}_n of the sphere with n marked points. We will always assume that n is at least three. In this setting there is an easy way to construct the moduli space \mathcal{M}_n . Any surface which is homeomorphic to the sphere is conformally isomorphic to the Riemann sphere by the Uniformization theorem. Hence for each point $(S, f) \in \mathcal{T}_n$ we can consider the associated vector $(z_1, z_2, \dots, z_n) \in \mathbb{P}^n$ of complex coordinates of the n marked points on the corresponding surface S . This vector is defined uniquely up to the action of the group of Möbius transformations. In particular, we can assume that the first three coordinates of any such vector are $0, 1$ and ∞ (or any other three distinct numbers for this purpose). Obviously, any vector $(0, 1, \infty, z_4, \dots, z_n) \in \mathbb{P}^n$ with all coordinates distinct is associated to some point in \mathcal{T}_n . Hence we can identify \mathcal{M}_n with a very concrete submanifold of \mathbb{C}^n or

even \mathbb{C}^{n-3} .

2.6 Positive matrices

As in many fields of mathematics, this work uses classical results from matrix theory (see, for example, [Gan66]).

Definition 2.6.1. We say that a matrix M is *non-negative* (or *positive*) and write $M \geq 0$ (or $M > 0$) if all its entries are non-negative (or, respectively, positive). Same definition also applies for vectors.

More generally, we write $A \geq B$ (or $A > B$) if $A - B \geq 0$ (respectively, $A - B > 0$).

Definition 2.6.2. A square matrix M is *reducible* if there exists a permutation matrix P such that conjugation by P puts M in the block form

$$P^{-1}MP = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}, \quad (2.1)$$

where M_{11} and M_{22} are square matrices. If no such permutation exists, the matrix M is *irreducible*.

It is obvious that any positive matrix is irreducible. Conjugating by a permutation matrix, any matrix can be written in the form

$$P^{-1}MP = \begin{pmatrix} M_{11} & 0 & \dots & 0 \\ M_{21} & M_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ M_{k1} & M_{k2} & \dots & M_{kk} \end{pmatrix},$$

where all blocks M_{ii} are square and irreducible. Clearly, the spectral radius of M is the maximum of spectral radii of M_{kk} .

The following theorem has numerous implications in many subjects.

Theorem 2.6.1 (Perron-Frobenius). *If M is an irreducible non-negative square matrix then there exist a unique largest eigenvalue (the Perron-Frobenius or leading eigenvalue) $\lambda(M)$ which is real and positive, and a unique up to scale positive eigenvector with eigenvalue $\lambda(M)$.*

Corollary 2.6.2. *If M is a non-negative square matrix then there exist a largest eigenvalue $\lambda(M)$ which is real and positive.*

Proof. The statement follows immediately from the previous theorem and preceding remark. \square

Definition 2.6.3. A matrix M is called *primitive* if there exists a power $k \in \mathbb{N}$ for which M^k is positive.

Denote by I_n the $n \times n$ identity matrix. The following is a well-known result.

Theorem 2.6.3. *If M is an irreducible non-negative $n \times n$ matrix then $(I_n + M)^{n-1} > 0$. In particular, the matrix $I_n + M$ is primitive.*

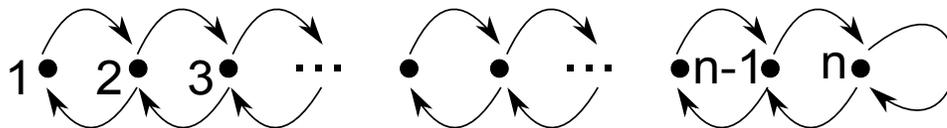
We prove a slightly more general statement.

Proposition 2.6.4. *If M is an irreducible non-negative $n \times n$ matrix and at least one diagonal entry of M is positive then $M^{2n-2} > 0$ and, hence, M is primitive.*

Proof. Without loss of generality, we can assume that all non-zero entries of M are equal to 1. Construct a directed graph G with n vertices using M as an adjacency matrix, i.e. adding an edge from i -th to j -th vertex if and only if the corresponding entry m_{ij} is equal to 1. Since M is irreducible, there exists a directed path in G between any two vertices. Indeed, take any vertex a and denote by A the set of all vertices you can reach starting at a . If A is not the whole set of vertices then a permutation, that puts all vertices in A before the rest of the vertices, conjugates M to a matrix in the block form (2.1). Note that the shortest path between any two vertices is evidently no longer than $n - 1$.

We write $M^k = (m_{ij}^k)$. We notice that m_{ij}^k is equal to the number of paths in G of length exactly k that start at i -th vertex and end at j -th vertex. Therefore, it is enough to prove that between any two points there exists a path of length $2n - 2$. Recall that M has a non-zero diagonal entry which corresponds to a loop in G at some vertex v . Any two vertices a and b can be connected by a path of length at most $2n - 2$ that passes through v because there exist paths of length at most $2n - 2$ connecting a to v and v to a . To construct a path of length $2n - 2$ we simply insert the loop at v an appropriate number of times into the former path. \square

The following example shows that $2n - 2$ is the minimal possible number in the last statement. Consider the matrix that has 1's on the diagonals $(k, k + 1)$ and

Figure 2.1: The graph with adjacency matrix M

$(k, k - 1)$, and also in the entry (n, n) :

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}.$$

The graph corresponding to M see in Figure 2.1. Every path of odd length from the first vertex to itself must pass through the loop at n -th vertex and, thus, has length at least $2n - 1$. Similarly, every path of even length from the first to the second vertex also must pass through the loop at n -th vertex and has length at least $2n - 2$. It follows, that for any $k < 2n - 2$ either m_{00}^k or m_{01}^k is equal to zero.

We will also use the following statement (compare to Section XIII.5 in [Gan66]).

Theorem 2.6.5. *For any irreducible matrix M there exists a power $k \in \mathbb{N}$ such that M^k , conjugating by an appropriate permutation P can be written in the block diagonal form*

$$P^{-1}M^kP = \begin{pmatrix} M_{11} & 0 & \dots & 0 \\ 0 & M_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & M_{kk} \end{pmatrix},$$

where all M_{ii} are positive and have the same leading eigenvalue as M^k .

Chapter 3

Background

3.1 Basic definitions

The main setup is the same as in [DH93].

Let f be an orientation-preserving branched self-cover of degree $d_f \geq 2$ of the 2-dimensional sphere \mathbb{S}^2 . The *critical set* Ω_f is the set of all points z in \mathbb{S}^2 where the local degree of f is greater than 1. The *postcritical set* \mathcal{P}_f is the union of all forward orbits of Ω_f , i.e. $\mathcal{P}_f = \cup_{i \geq 1} f^i(\Omega_f)$. A branched cover f is called *postcritically finite* if \mathcal{P}_f is finite. More generally, a pair (f, P_f) of a branched cover $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ and a finite set $P_f \subset \mathbb{S}^2$ is called a *Thurston map* if P_f is forward invariant and contains all critical values of f (and, hence, contains \mathcal{P}_f). Denote $p_f = \#P_f$. Note that extra marked points provide additional combinatorial information about f but do not change topological structure of f since that is determined by critical points and critical values.

Two Thurston maps f and g are *Thurston equivalent* if and only if there exist two homeomorphisms $h_1, h_2: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that the diagram

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{S}^2, P_g) \\ f \downarrow & & \downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_2} & (\mathbb{S}^2, P_g) \end{array}$$

commutes, $h_1|_{P_f} = h_2|_{P_f}$, and h_1 and h_2 are homotopic relative to P_f .

A simple closed curve γ is called *essential* if every component of $\mathbb{S}^2 \setminus \gamma$ contains at least two points of P_f . We consider essential simple closed curves up to free homotopy in $\mathbb{S}^2 \setminus P_f$. A *multicurve* is a finite set of pairwise disjoint and non-homotopic

essential simple closed curves. Denote by $f^{-1}(\Gamma)$ the multicurve consisting of all essential preimages of curves in Γ . A multicurve $\Gamma = (\gamma_1, \dots, \gamma_n)$ is called *invariant* if each component of $f^{-1}(\gamma_i)$ is either non-essential, or it is homotopic (in $\mathbb{S}^2 \setminus P_f$) to a curve in Γ (i.e. $f^{-1}(\Gamma) \subseteq \Gamma$). We say that Γ is *completely invariant* if $f^{-1}(\Gamma) = \Gamma$.

Every multicurve Γ has its associated *Thurston matrix* $M_\Gamma = (m_{i,j})$ with

$$m_{i,j} = \sum_{\gamma_{i,j,k}} (\deg f|_{\gamma_{i,j,k}} : \gamma_{i,j,k} \rightarrow \gamma_j)^{-1}$$

where $\gamma_{i,j,k}$ ranges through all preimages of γ_j that are homotopic to γ_i . Since all entries of M_Γ are non-negative real, the leading eigenvalue λ_Γ of M_Γ is real and non-negative (see Corollary 2.6.2). Note that to define the Thurston matrix M_Γ , we do not require that Γ be invariant.

A multicurve Γ is a *Thurston obstruction* if $\lambda_\Gamma \geq 1$. A Thurston obstruction Γ is *minimal* if no proper subset of Γ is itself an obstruction. We call Γ a *simple obstruction* (compare [Pil01]) if no permutation of the curves in Γ puts M_Γ in the block form

$$M_\Gamma = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix},$$

where the leading eigenvalue of M_{11} is less than 1. If such a permutation exists, it follows that M_{22} is a Thurston matrix of an invariant multicurve with the same leading eigenvalue as M_Γ . It is thus evident that every obstruction contains a simple one. The following is an exercise in linear algebra.

Proposition 3.1.1. *A multicurve Γ is a simple obstruction if and only if there exists a vector $v > 0$ such that $M_\Gamma v \geq v$.*

Proof. Let v be a non-negative vector such that $M_\Gamma v \geq v$ with a maximal possible number of positive components. Applying a permutation if necessary, we assume that $v = (0, v_1)^T$ where $v_1 > 0$. If we write M_Γ in the corresponding block form we get:

$$M_\Gamma = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}.$$

If the leading eigenvalue of M_{11} is greater than 1, then there exists a non-negative eigenvector v_2 for M_{11} such that $M_\Gamma(v_2 v_1)^T \geq (v_2 v_1)^T$. The choice of $v = (0, v_1)^T$ implies that either v is positive, or the leading eigenvalue of M_{11} is less than 1, and Γ is not a simple obstruction.

On the other hand, it is clear that if there exists a positive vector v with $Mv \geq v$,

then the leading eigenvalue of M is at least 1. Therefore, if there exists a positive vector v with $M_\Gamma v \geq v$, then M_Γ can not be written in a block form as above. \square

Note that every minimal obstruction is simple, and that a union of two disjoint simple obstructions is also simple. A *Levy cycle* is a multicurve $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ such that for every $i = \overline{1, n}$, there exists a preimage component of γ_i that is homotopic to γ_{i+1} and is mapped to γ_i by f with degree 1 (we set $\gamma_{n+1} = \gamma_1$). Every Levy cycle is a Thurston obstruction.

Thurston's original characterization theorem is formulated as follows:

Theorem 3.1.2 (Thurston's Theorem [DH93]). *A postcritically finite branched cover $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ with hyperbolic orbifold is either Thurston-equivalent to a rational map g (which is then necessarily unique up to conjugation by a Moebius transformation), or f has a Thurston obstruction.*

Remark 3.1.3. In the original formulation in [DH93], a Thurston obstruction was required to be invariant. Our formulation is both weaker in one direction and stronger in the other direction. However, in our proof we show that if there exists a Thurston obstruction for f , then there also exists a simple invariant obstruction (see Corollary 4.3.3 and Proposition 4.3.4).

General rigorous definition of orbifolds and their Euler characteristic can be found in [Mil06a]. In our case, there is a unique and straightforward way to construct the minimal function v_f of all functions $v: \mathbb{S}^2 \rightarrow \mathbb{N} \cup \{\infty\}$ satisfying the following two conditions:

- (i) $v(x) = 1$ when $x \notin P_f$;
- (ii) $v(x)$ is divisible by $v(y) \deg_y f$ for all $y \in f^{-1}(x)$.

We say that f has hyperbolic orbifold $O_f = (\mathbb{S}^2, v_f)$ if the Euler characteristic of O_f

$$\chi(O_f) = 2 - \sum_{x \in P_f} \left(1 - \frac{1}{v_f(x)} \right) \quad (3.1)$$

is less than 0, and parabolic orbifold otherwise. We discuss Thurston maps with parabolic orbifolds in more detail in Section 3.5.

3.2 Teichmüller space and Thurston iteration

Let \mathcal{T}_f be the Teichmüller space modeled on the marked surface (\mathbb{S}^2, P_f) and \mathcal{M}_f be the corresponding moduli space. The space \mathcal{T}_f can be defined as the quotient of the space of all diffeomorphisms from (\mathbb{S}^2, P_f) to the Riemann sphere \mathbb{P} modulo post-composition with Moebius transformations and isotopies relative P_f . This is a $(p_f - 3)$ -dimensional complex manifold (in case $p_f \leq 3$ it is a one point set). We write $\tau = \langle h \rangle$ if a point τ is represented by a diffeomorphism h . Correspondingly, points of \mathcal{M}_f are represented by $h(P_f)$ modulo post-composition with Moebius transformations. Denote by $\pi : \mathcal{T}_f \rightarrow \mathcal{M}_f$ the canonical covering map which sends $h \mapsto h|_{P_f}$. The (pure) mapping class group of (\mathbb{S}^2, P_f) is canonically identified with the group of deck transformations of π . For more background on Teichmüller spaces see, for example, [IT92, Hub06].

When we talk about a Riemann surface R and a set of points P on it, we endow it by default with the Poincaré metric with constant curvature -1 of $R \setminus P$ provided the latter surface is hyperbolic, which will always be the case in this work.

The cotangent space at a point $\tau = \langle h \rangle$ in the Teichmüller space \mathcal{T}_f is canonically isomorphic to the space of all meromorphic quadratic differentials $Q(\mathbb{P}, h(P_f))$ on the marked Riemann surface corresponding to τ (which is $(\mathbb{P}, h(P_f))$) that are holomorphic on $\mathbb{P} \setminus h(P_f)$ and have at most simple poles in $h(P_f)$. The Teichmüller and Weil-Petersson norms for $q \in Q(\mathbb{P}, h(P_f))$ are defined as follows:

$$\|q\|_T = 2 \int_{\mathbb{P}} |q|$$

and

$$\|q\|_{WP} = \left(\int_{\mathbb{P}} \rho^{-2} |q|^2 \right)^{1/2},$$

where ρ is the hyperbolic distance element of $(\mathbb{P}, h(P_f))$. The duals of these two norms define Finsler metrics on \mathcal{T}_f (the metric defined by the Weil-Petersson norm is not only Finsler but Hermitian). We write $d_T(\cdot, \cdot)$ and $d_{WP}(\cdot, \cdot)$ for the distances between points in the Teichmüller space with respect to the corresponding metric. We will measure distance with respect to both metrics in order to prove Theorem 4.5.2. For this we will need the following estimates [McM00, Proposition 2.4 and Theorem 4.4]:

Proposition 3.2.1. *i. For any tangent vector v to \mathcal{T}_f we have*

$$\|v\|_{WP} \leq C^0 \|v\|_T.$$

ii. If $l(\gamma, \tau) > \varepsilon$ for all essential simple closed curves γ then for any tangent vector v to \mathcal{T}_f at τ we have

$$\|v\|_{WP} \geq C^1(\varepsilon)\|v\|_T$$

where $C^1(\varepsilon)$ is a constant depending on ε .

Consider an essential simple closed curve γ in (\mathbb{S}^2, P_f) . For each complex structure τ on (\mathbb{S}^2, P_f) , there exists a unique geodesic γ_τ in the homotopy class of γ . As above, we denote by $l(\gamma, \tau)$ the length of the geodesic γ_τ homotopic to γ on the Riemann surface corresponding to $\tau \in \mathcal{T}_f$. This defines a continuous function from \mathcal{T}_f to \mathbb{R}_+ for any given γ . Moreover, $\log l(\gamma, \tau)$ is a Lipschitz function with Lipschitz constant 1 with respect to the Teichmüller metric (see [Hub06, Theorem 7.6.4]; note that in [DH93, Proposition 7.2] the constant is 2 because of a different normalization of the Teichmüller metric). We will use the same notation $l(\gamma, R)$ for the hyperbolic length of a curve γ in a hyperbolic surface R . Recall that the length of a simple closed geodesic γ on a hyperbolic Riemann surface R is closely related to the supremum $M(\gamma, R)$ of moduli of all annuli on R that are homotopic to this geodesic (see Theorem 2.4.3), namely

$$\frac{\pi}{l(\gamma, R)} - 1 < M(\gamma, R) < \frac{\pi}{l(\gamma, R)}.$$

We define the key player in the proof of Theorem 3.1.2 — the Thurston pullback σ_f — as follows. Suppose $\tau \in \mathcal{T}_f$ is represented by a homeomorphism h_τ . Consider the following diagram:

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & & \\ \downarrow f & & \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_\tau} & (\mathbb{P}, h_\tau(P_f)) \end{array} \quad (3.2)$$

We can pull back the standard complex structure μ_0 on \mathbb{P} to an almost-complex structure $f^*h_\tau^*\mu_0$ on (\mathbb{S}^2, P_f) . By the Measurable Riemann mapping theorem (Theorem 2.3.1), it induces a complex structure on (\mathbb{S}^2, P_f) . Let h_1 be a conformal isomorphism between (\mathbb{S}^2, P_f) endowed with the complex structure $f^*h_\tau^*\mu_0$ and \mathbb{P} . Set $\sigma_f(\tau) = \tau_1$ where τ_1 is the point represented by h_1 .

Now we can complete the previous diagram by setting $f_\tau = h_\tau \circ f \circ h_1^{-1}$ so that it commutes:

$$\begin{array}{ccc}
(\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{P}, h_1(P_f)) \\
\downarrow f & & \downarrow f_\tau \\
(\mathbb{S}^2, P_f) & \xrightarrow{h_\tau} & (\mathbb{P}, h_\tau(P_f))
\end{array} \tag{3.3}$$

Note that from definition of f_τ , it follows that f_τ respects the standard complex structure μ_0 and, hence, is rational. When we choose a representing homeomorphism h_τ , we have the freedom to post-compose h_τ with any Moebius transformation; similarly, the choice of h_τ defines h_1 up to a post-composition by Moebius transformation. Thus, f_τ is defined up to pre- and post-composition by Moebius transformations.

It has been shown in [DH93] that σ_f is a holomorphic self-map of \mathcal{T}_f and that the co-derivative of σ_f satisfies $(d\sigma_f(\tau))^* = (f_\tau)_*$ where $(f_\tau)_*$ is the push-forward operator on quadratic differentials. It is straightforward to prove $\|d\sigma_f\|_T = \|(d\sigma_f)^*\|_T \leq 1$, and with a little more effort one gets $\|(d\sigma_f^k)\|_T < 1$, for some $k \in \mathbb{N}$, when f has hyperbolic orbifold (see [BGL]), hence, σ_f is weakly contracting on \mathcal{T}_f with respect to the Teichmüller metric for any such f . Since \mathcal{T}_f is path-connected, it follows that σ_f has at most one fixed point if f has hyperbolic orbifold, and every forward orbit of σ_f converges to the fixed point in the case there exists one.

The following proposition [DH93, Proposition 2.3] relates dynamical properties of σ_f to the original question.

Proposition 3.2.2. *A Thurston map f is equivalent to a rational function if and only if σ_f has a fixed point.*

Moreover, it is not hard to see that non-conjugate rational functions must correspond to different fixed points. Therefore, the uniqueness part of Theorem 3.1.2 is clear.

The *canonical* obstruction Γ_f is the set of all homotopy classes of curves γ that satisfy $l(\gamma, \sigma_f^n(\tau)) \rightarrow 0$ for all (or, equivalently, for some) $\tau \in \mathcal{T}_f$. The following theorems are due to Kevin Pilgrim [Pil01]. We give alternative proofs of these statements below.

Theorem 3.2.3 (Canonical Obstruction Theorem). *If for a Thurston map with hyperbolic orbifold its canonical obstruction is empty then it is Thurston equivalent to a rational function. If the canonical obstruction is not empty then it is a Thurston obstruction.*

Theorem 3.2.4 (Curves Degenerate or Stay Bounded). *For any point $\tau \in \mathcal{T}_f$ there exists a bound $L = L(\tau, f) > 0$ such that for any essential simple closed curve $\gamma \notin \Gamma_f$ the inequality $l(\gamma, \sigma_f^n(\tau)) \geq L$ holds for all n .*

Remark 3.2.5. In the terms to be defined in the next section, the previous two theorems can be reformulated as follows. Note that this version does not require that Thurston map f has hyperbolic orbifold.

Theorem 3.2.6. *The accumulation set of $\pi(\sigma_f^n(\tau))$ in the compactified moduli space $\overline{\mathcal{M}}_f$ is a compact subset of $\mathcal{S}_{[\Gamma_f]}$. If Γ_f is not empty then the sequence $\sigma_f^n(\tau)$ tends to \mathcal{S}_{Γ_f} .*

3.3 The augmented Teichmüller space

Very relevant and good surveys on augmented Teichmüller spaces can be found in [Wol03, Wol09]. Here we remind the reader of the basics that we will need later on.

Let S be a topological hyperbolic surface of finite type (i.e., a surface of genus g with n punctures, where $\chi(S) = 2 - 2g - n < 0$). Recall that the Teichmüller space of S is the space of all marked Riemann surfaces of the same type as S . Each point in $\mathcal{T}(S)$ can be represented by a homeomorphism between S and a Riemann surface. One defines the augmented Teichmüller space $\overline{\mathcal{T}}(S)$ as the space of all stable marked Riemann surfaces *with nodes* of the same type as S . The *type* of a noded surface is defined by its topological genus (more precisely, by the genus of a topological surface one obtains by opening up all nodes) and the number of marked points (excluding nodes). In our case, the case of the sphere with marked points, any *surface with nodes* and marked points R (or a *noded surface*) is a collection of *components* that are topological spheres with marked points, so that two components intersect in at most one marked point, each marked point belongs to at most two components, and the union of all components is connected and simply connected. The marked points that belong to two components of R are called *nodes*. Any component of a noded surface R can be obtained from a connected component of the complement of nodes in R by adding to it all incident nodes as marked points. The genus, in this setting, is always 0 thus the type of such a noded surface is determined by the number of marked points that are not nodes. *Stable* noded surfaces are those for which every component is hyperbolic. We represent points in $\overline{\mathcal{T}}(S)$ not only by homeomorphisms but also by continuous maps from S to a noded Riemann surface that are allowed to send a whole simple closed curve (or, which is the same up to homotopy, a closed annulus) in the complement of marked points to a node. In other words, we allow to

pinch some of the curves on S into nodes. The same idea is used to construct $\overline{\mathcal{M}}(S)$ — the Bers compactification of the moduli space [Ber74b, Ber74a]. The canonical projection from $\mathcal{T}(S)$ to $\mathcal{M}(S)$ extends to the canonical projection from $\overline{\mathcal{T}}(S)$ to $\overline{\mathcal{M}}(S)$.

The augmented Teichmüller space $\overline{\mathcal{T}}_f$ is a stratified space with strata corresponding to different multicurves on (\mathbb{S}^2, P_f) . We denote by \mathcal{S}_Γ the stratum corresponding to the multicurve Γ , i.e., the set of all noded surfaces for which the nodes come from pinching all elements of Γ and there are no other nodes. In particular, $\mathcal{T}_f = \mathcal{S}_\emptyset$. Strata of $\overline{\mathcal{M}}_f$ are labeled by equivalence classes $[\Gamma]$ of multicurves, where two multicurves Γ_1 and Γ_2 are in the same class if and only if one can be transformed to the other by an element of the mapping class group or, equivalently, if the respective elements of Γ_1 and Γ_2 separate points of P_f in the same way. We naturally denote $\partial\mathcal{T}_f = (\overline{\mathcal{T}}_f \setminus \mathcal{T}_f)$.

Each point in the stratum \mathcal{S}_Γ is a collection of complex structures on the components of the corresponding topological noded surface with marked points. Therefore, \mathcal{S}_Γ is the product of Teichmüller spaces of these components. We will refer to the points in the Teichmüller spaces of components as *coordinates* of a point in \mathcal{S}_Γ . Within each stratum one can define its own natural Teichmüller (as the ∞ -product of Teichmüller metrics of components) or Weil-Petersson (as the 2-product of Teichmüller metrics of components) metrics. The following theorem [Mas76] shows the interplay between the notion of the augmented Teichmüller space and the Weil-Petersson metric.

Theorem 3.3.1. *The augmented Teichmüller space is homeomorphic to the completion of the Teichmüller space with respect to the Weil-Petersson metric. Moreover, the restriction of the completed Weil-Petersson metric to each stratum is the Weil-Petersson metric of this stratum.*

The following estimate on the Weil-Petersson norm of the coderivative of σ_f shows that σ_f is Lipschitz with respect to the Weil-Petersson metric on \mathcal{T}_f and hence extends to its completion $\overline{\mathcal{T}}_f$. This already proves Theorem 1.0.2.

Proposition 3.3.2. $\|(d\sigma_f)^*\|_{WP} \leq \sqrt{d_f}$.

Proof. Let us prove the statement for an arbitrary point $\tau = \langle h \rangle \in \mathcal{T}_f$. To simplify notation, set $g = f_\tau$, $d = d_f$ and $P = h(P_f)$. We need to prove then that $\|g_*q\|_{WP} \leq \sqrt{d}\|q\|_{WP}$ for any $q \in Q(\mathbb{P}, P')$ where P' is the image of P_f for $\sigma_f(\tau)$.

If we take some small domain $U \subset (\mathbb{P} \setminus P)$ with local coordinate ζ such that it has exactly d disjoint preimages $U_i, i = 1, \dots, d$, with $g_i: U \rightarrow U_i$ the local branches

of g^{-1} , then

$$g_*q|_U = \sum_i g_i^*q.$$

Let ρ^2 and ρ_1^2 stand for the hyperbolic area elements on $\mathbb{P} \setminus P$ and $\mathbb{P} \setminus P'$. The hyperbolic area element on $\mathbb{P} \setminus g^{-1}(P)$ is given by $g^*\rho^2$. The inclusion map $I: \mathbb{P} \setminus g^{-1}(P) \rightarrow \mathbb{P} \setminus P'$ is length-decreasing; therefore $\rho_1^2 \leq g^*\rho^2$.

Now we can locally estimate:

$$\int_U \frac{|g_*q|^2}{\rho^2} = \int_U \frac{|\sum_i g_i^*q|^2}{\rho^2} \leq d \sum_i \int_U \frac{|g_i^*q|^2}{\rho^2} = d \sum_i \int_{U_i} \frac{|q|^2}{g^*\rho^2} \leq d \int_{g^{-1}(U)} \frac{|q|^2}{\rho_1^2},$$

where the first inequality follows from the fact that

$$\left| \sum_{i=1}^d a_i \right|^2 \leq d \sum_{i=1}^d |a_i|^2.$$

Combining local estimates, we get

$$\|g_*q\|_{WP} \leq \sqrt{d}\|q\|_{WP},$$

as required. \square

In Section 4.1 we refine the statement of Theorem 1.0.2 by showing that a certain extension of σ_f to the boundary of $\overline{\mathcal{T}}_f$, defined in terms of pullbacks of complex structures on noded Riemann surfaces, is continuous (see Theorem 4.1.4) and, hence, coincides with the extension given by metric completion.

Note that $l(\gamma, \tau)$ extends continuously to a function from $\overline{\mathcal{T}}_f$ to $[0, +\infty]$. A curve of length zero corresponds to a node; a curve of infinite length has to pass through at least one node (it has positive intersection number with at least one curve of length zero). By definition, $\tau \in \mathcal{S}_\Gamma$ when $l(\gamma, \tau) = 0$ for those and only those homotopy classes of curves γ that belong to Γ .

We can also view l as a map $l: \overline{\mathcal{T}}_f \rightarrow [0, +\infty]^{\mathcal{H}}$ where \mathcal{H} is the set of all homotopy classes of essential simple closed curves. We endow $[0, +\infty]^{\mathcal{H}}$ with the product topology.

Proposition 3.3.3. *The map $l: \overline{\mathcal{T}}_f \rightarrow [0, +\infty]^{\mathcal{H}}$ is an embedding.*

Proof. We know that l is continuous on $\overline{\mathcal{T}}_f$ and that $l(\mathcal{T}_f)$ is mapped onto its image by a homeomorphism (compare Theorems 3.12 and 3.15 in [IT92]). Similarly we

can see, using the product structure of the boundary strata, that each of them is mapped to its image by a homeomorphism. Evidently images of different strata are disjoint. Hence the map is injective. It is straightforward to check that its inverse is also continuous. \square

In other words, the homotopy classes of nodes and the lengths of all simple closed geodesics uniquely define a point in $\overline{\mathcal{T}}_f$ and the topology on $\overline{\mathcal{T}}_f$ can be defined using the topology of $[0, +\infty]^{\mathcal{H}}$.

3.4 Technical background

First we prove the following technical propositions.

Proposition 3.4.1. *Let X be an open hyperbolic subset of the Riemann sphere and p be an isolated puncture of X . Take a nested sequence $\{U_n\}$ of closed neighborhoods of p . Denote by ρ_X the hyperbolic distance element on X and ρ_n the hyperbolic distance element on the set $X \setminus U_n$. If $\bigcap U_n = \{p\}$ then $\{\rho_n(x)\}$ tends to $\rho_X(x)$ for any point $x \in X$. Moreover, the convergence is uniform on compact subsets of X .*

Proof. The identity inclusion of $X \setminus U_n$ into $X \setminus U_m$ for $n < m$ is obviously holomorphic, hence length-decreasing; the same is true for inclusions of $X \setminus U_n$ into X . Therefore, $\{\rho_n(x)\}$ is decreasing and the limit is greater or equal than $\rho_X(x)$.

Without loss of generality we assume that $p = \infty$. We know that $\rho_X(x) = 2/\sup |f'(0)|$ where f runs through the set of all holomorphic maps from the unit disk \mathbb{D} to X that send 0 to x (this is the definition of the Kobayashi metric which is well known to coincide with the hyperbolic metric for hyperbolic Riemann surfaces). Let $\varepsilon > 0$. Pick such a map $f: \mathbb{D} \rightarrow X$ so that $2/|f'(0)| < (1 + \varepsilon)\rho_X$. Set $D_t = \{|z| \leq t, z \in \mathbb{C}\}$. We notice that $f(D_{1-\varepsilon})$ is compact in \mathbb{C} , hence bounded. Then $f(D_{1-\varepsilon}) \subset X \setminus U_n$ for all n large enough. Hence $f_\varepsilon(z) := f((1 - \varepsilon)z)$ is a holomorphic map from \mathbb{D} to $X \setminus U_n$ with $f_\varepsilon(0) = x$ and $f'_\varepsilon(0) = (1 - \varepsilon)f'(0)$. Thus by definition

$$\rho_n(x) \leq \frac{2}{f'_\varepsilon(0)} = \frac{2}{(1 - \varepsilon)f'(0)} \leq \frac{1 + \varepsilon}{1 - \varepsilon} \rho_X(x).$$

This shows that $\rho_n \rightarrow \rho_X$. Uniform convergence on compact subsets of X is evident. \square

Proposition 3.4.2. *Let $\{U_n\}$ be an increasing nested sequence of open subsets of \mathbb{P}^1 such that the complement of the union U of all U_n consists of finitely many points*

and $\{f_n: \mathbb{P} \rightarrow \mathbb{P}\}$ be a sequence of continuous maps that fix three points a_1, a_2, a_3 on \mathbb{P} . If f_n^{-1} is well-defined (i.e. every point has exactly one preimage) and conformal on U_n for all n , then $\{f_n\}$ converges uniformly to the identity mapping on \mathbb{P} .

Proof. Let $V_n = f_n^{-1}(U_n)$. Suppose first that the three points a_1, a_2, a_3 are inside V_n for all n . Consider the sequence of conformal inverses $\{f_n^{-1}: U_n \rightarrow V_n\}$. All of these functions are conformal on $U_1 \setminus \{a_1, a_2, a_3\}$ and do not assume values $\{a_1, a_2, a_3\}$. Hence, by Montel's theorem, $\{f_n^{-1}\}$ forms a normal family on $U_1 \setminus \{a_1, a_2, a_3\}$ and we can choose a subsequence $\{f_{i(1,n)}^{-1}\}$ converging locally uniformly on U_1 to a conformal map. By the same reasoning, we choose a subsequence $\{f_{i(2,n)}^{-1}\}$ of $\{f_{i(1,n)}^{-1}\}$ that converges locally uniformly on U_2 and so on. Then the diagonal subsequence $\{f_{i(m,m)}^{-1}\}$ converges locally uniformly on U . Since the limit is conformal on \mathbb{P} except for finitely many points, it is in fact conformal on the whole sphere, and since it fixes a_1, a_2, a_3 it is the identity. Note that the same reasoning implies that we can choose a subsequence converging locally uniformly to the identity from any subsequence of the sequence $\{f_n^{-1}\}$. Therefore $\{f_n^{-1}\}$ converges locally uniformly to the identity on U . Since every point in U_n has a unique f_n -preimage, it is easy to see that $\{f_n\}$ converges uniformly to the identity as required.

We now consider the general case when a_1, a_2, a_3 are chosen arbitrarily. Pick points b_1, b_2, b_3 in U_1 and set $c_i^n = f_n^{-1}(b_i)$ for $i = \overline{1, 3}$ so that $c_i^n \in V_n$ for all n . Set $g_n = f_n \circ \phi_n$ where ϕ_n is the Moebius transformation that sends b_i to c_i^n . By the earlier arguments, $\{g_n\}$ uniformly converges to the identity. In particular, $g_n(\phi_n^{-1}(a_i)) = f_n(a_i) = a_i$ implies that $\phi_n^{-1}(a_i) \rightarrow a_i$ for $i = \overline{1, 3}$. Therefore, $\{\phi_n^{-1}\}$ converges uniformly to the identity and the general case follows. \square

3.5 Thurston maps with parabolic orbifolds

A complete classification of postcritically finite branched covers (i.e. Thurston maps (f, P_f) , where P_f is equal to the postcritical set \mathcal{P}_f) with parabolic orbifolds has been given in [DH93]. All rational functions that are postcritically finite branched covers with parabolic orbifold has been extensively described in [Mil06b]. However, no classification has been developed yet for general Thurston maps. In this section, we remind the reader of basic results on Thurston maps with parabolic orbifolds.

Recall that a map $f: (S_1, v_1) \rightarrow (S_2, v_2)$ is a covering map of orbifolds if $v_1(x) \deg_x f = v_2(f(x))$ for any $x \in S_1$. The following proposition from [DH93] is crucial.

Proposition 3.5.1. *i. If $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is a postcritically finite branched cover, then $\chi(O_f) \leq 0$.*

ii. If $\chi(O_f) = 0$, then $f: O_f \rightarrow O_f$ is a covering map of orbifolds.

Equation (3.1) gives six possibilities for $\chi(O_f) = 0$. If we record all the values of v_f that are bigger than 1, we get one of the following orbifold signatures.

- (∞, ∞) ,
- $(2, 2, \infty)$,
- $(2, 4, 4)$,
- $(2, 3, 6)$,
- $(3, 3, 3)$,
- $(2, 2, 2, 2)$.

We are mostly interested in the last case. A $(2, 2, 2, 2)$ -map is a Thurston map that has orbifold with signature $(2, 2, 2, 2)$. From now on in this section, we always assume that f is a $(2, 2, 2, 2)$ -map. An orbifold with signature $(2, 2, 2, 2)$ is a quotient of a torus T by an involution i ; the four fixed points of the involution i correspond to the points with ramification weight 2 on the orbifold. Denote by p the corresponding branched cover from T to \mathbb{S}^2 ; it has exactly 4 simple critical points which are the fixed points of i . It follows that f can be lifted to a covering self-map \hat{f} of T (see [DH93]).

Take any simple closed curve γ on $\mathbb{S}^2 \setminus \mathcal{P}_f$. Then $p^{-1}(\gamma)$ has either one or two components that are simple closed curves.

Proposition 3.5.2. *If there are exactly two postcritical points of f in each complementary component of γ , then the p -preimage of γ consists of two components that are homotopic in T and non-trivial in $H_1(T, \mathbb{Z})$. Otherwise, all preimages of γ are trivial.*

Proof. Note that postcritical points of f are, by definition, the fixed points of involution i and critical values of p . Since γ separates \mathbb{S}^2 into two connected components, if γ has exactly one preimage α , then α must separate T into two components and, thus, be contractible. The disk bounded by α in T is mapped to one of the connected components of the complement of γ with degree two. Therefore, there exists

exactly one critical point of p is this disk, and, hence, exactly one postcritical point in its image.

If γ has two preimages, then they do not intersect, which implies that they are homotopic to each other or at least one of them is contractible. However, if one of the preimages is contractible, the other one also is, since they are mapped to each other by i . In this case, the disk bounded by a contractible preimage component is mapped by f one-to-one to a component of the complement of γ . Thus, this complementary component has no postcritical points.

The last case is when γ has two preimages that are homotopic to each other and are not trivial in $H_1(T, \mathbb{Z})$. Then the complement of the full preimage of γ has two components that are annuli that are mapped by p to complementary components of γ with degree two. It follows that there are exactly two postcritical points in each of these components. \square

Every homotopy class of simple closed curves γ on T defines, up to sign, an element $\langle \gamma \rangle$ of $H_1(T, \mathbb{Z})$. If a simple closed curve γ on $\mathbb{S}^2 \setminus \mathcal{P}_f$ has two p -preimages, then they are homotopic by the previous proposition. Therefore, every homotopy class of simple closed curves γ on $\mathbb{S}^2 \setminus \mathcal{P}_f$ also defines, up to sign, an element $\langle \gamma \rangle$ of $H_1(T, \mathbb{Z})$. It is clear that for any $h \in H_1(T, \mathbb{Z})$ there exists a homotopy class of simple closed curves γ such that $h = n\langle \gamma \rangle$ for some $n \in \mathbb{Z}$.

Since $H_1(T, \mathbb{Z}) \cong \mathbb{Z}^2$, the push-forward operator \hat{f}_* is a linear operator. It is easy to see that the determinant of \hat{f}_* is equal to the degree of \hat{f} , which is in turn equal to the degree of f . Existence of invariant multicurves for f is related to the action of \hat{f}_* on $H_1(T, \mathbb{Z})$.

Proposition 3.5.3. *Suppose that a component γ' of the f -preimage of a simple closed curve γ on $\mathbb{S}^2 \setminus \mathcal{P}_f$ is homotopic γ . Take a p -preimage α of γ . Then $\hat{f}_*(\langle \alpha \rangle) = \pm d\langle \alpha \rangle$, where d is the degree of f restricted to γ' .*

Proof. By the previous proposition, if γ is not essential in $\mathbb{S}^2 \setminus \mathcal{P}_f$ then $\langle \alpha \rangle = 0$ and the claim holds. Otherwise, both γ and γ' have exactly two components in their p -preimages. Denote by α' a component of the p -preimage of γ' that is mapped to α by \hat{f} with degree d . A homotopy between γ and γ' lifts to a homotopy between α and α' on T . Therefore, α and α' define the same (up to sign) element of $H_1(T, \mathbb{Z})$, i.e. $\langle \alpha' \rangle = \pm \langle \alpha \rangle$. Since $\hat{f}(\alpha') = \alpha$, we get that $\hat{f}_*(\langle \alpha \rangle) = \hat{f}_*(\pm \langle \alpha' \rangle) = \pm d\langle \alpha \rangle$. \square

More generally, we obtain the following.

Proposition 3.5.4. *Let γ be a simple closed curve on $\mathbb{S}^2 \setminus \mathcal{P}_f$ such that there are two points of the postcritical set \mathcal{P}_f in each complementary component of γ . If all components of the f -preimage of γ have zero intersection number with γ in $\mathbb{S}^2 \setminus \mathcal{P}_f$, then $\hat{f}_*(\langle \gamma \rangle) = \pm d \langle \gamma \rangle$, where d is the degree of f restricted to any preimage of γ .*

Proof. Let γ' be a component of the f -preimage of γ . As before, denote by α' a component of the p -preimage of γ' that is mapped by \hat{f} to a component α of the p -preimage of γ with some degree d . Then $\hat{f}_*(\langle \alpha' \rangle) = \pm d \langle \alpha \rangle$. Since γ and γ' have zero intersection number in $\mathbb{S}^2 \setminus \mathcal{P}_f$, the homotopy classes of α and α' in T also have zero intersection number. By Proposition 3.5.2, $\langle \alpha \rangle$ is non-zero in $H_1(T, \mathbb{Z})$; the last equality implies that $\langle \alpha' \rangle$ is also non-zero in $H_1(T, \mathbb{Z})$. It follows that $\langle \alpha' \rangle = \pm \langle \alpha \rangle$. \square

Chapter 4

Thurston's theorem for rational maps

4.1 Extension of σ_f to the augmented Teichmüller space

Let $\tau = \langle h_\tau \rangle \in \mathcal{S}_\Gamma$ where $\Gamma \neq \emptyset$. Let R be the Riemann surface with nodes corresponding to τ . We look again at the commutative diagram (3.2), only this time on the bottom-right we have not the Riemann sphere but a Riemann surface R with nodes and marked points that consists of several Riemann spheres touching at the nodes with the image of P_f on it (see the commutative diagram (4.2) below). Consider the full preimage $f^{-1} \circ h_\tau^{-1}(N)$ of the set of nodes N of R on \mathbb{S}^2 . We obtain a topological noded surface T^0 with nodes $f^{-1} \circ h_\tau^{-1}(N)$ which is not necessarily stable. For example, it is not stable if a component of $f^{-1}(\gamma)$ is non-essential for some node γ . We consider the stabilisation T^1 of T^0 which is defined as follows. Every non-hyperbolic component of (T^0, P_f) is a sphere with at most two marked points and nodes, hence is obtained by pinching either

- i. a simple closed curve that is non-essential in $\mathbb{S}^2 \setminus P_f$
- ii. or a pair of simple closed curves that are homotopic to each other in $\mathbb{S}^2 \setminus P_f$.

In both cases, these components have a unique possible complex structure, thus they do not carry any information, and we collapse each of them to a point to produce a stable noded surface (T^1, P_f) . In the first case, we obtain an ordinary point of T^1 if the pinched non-essential closed curve was null-homotopic and a marked point of T^1 otherwise. In the second case, we obtain a node of T^1 if the pinched curves are essential, otherwise we get an ordinary point or a marked point as in the first case.

Note that with this construction several adjacent components might be pinched to a single point.

Denote by T the topological noded surface model of R . In other words, T is R viewed as a topological surface. Let $\text{id}: R \rightarrow T$ be the canonical homeomorphism between the two surfaces. Let $\tilde{h} = \text{id} \circ h_\tau$ be the canonical projection map $\tilde{h}: (\mathbb{S}^2, P_f) \rightarrow T$; we see that \tilde{h} sends any connected component of $h_\tau^{-1}(N)$ to a point and maps any connected component of $\mathbb{S}^2 \setminus h_\tau^{-1}(N)$ homeomorphically onto a component of T without finitely many points. Similarly, let \tilde{h}_1 be the canonical projection map $\tilde{h}_1: (\mathbb{S}^2, P_f) \rightarrow T^1$ that sends any connected component of $f^{-1} \circ h_\tau^{-1}(N)$ to a point; any connected component of $\mathbb{S}^2 \setminus f^{-1} \circ h_\tau^{-1}(N)$ is mapped by \tilde{h}_1 to a point, if that component is non-hyperbolic, otherwise it is homeomorphically mapped onto a component of T^1 without finitely many points. The maps \tilde{h} and \tilde{h}_1 are evidently injective on P_f .

Each component C_i^1 of T^1 is mapped to some component C_j of T by a branched cover $f^{C_i^1}$ that can be defined using the following diagram which commutes on each component of T^1 .

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{\tilde{h}_1} & (T^1, \tilde{h}_1(P_f)) \\ f \downarrow & & \downarrow \{f^{C_i^1}\} \\ (\mathbb{S}^2, P_f) & \xrightarrow{\tilde{h}} & (T, \tilde{h}(P_f)) \end{array}$$

We define maps $h_\tau^{C_j}: C_j \rightarrow R_j$, where R_j are the corresponding components of R , to be the unique maps that make the following diagram commute:

$$\begin{array}{ccc} & (T, \tilde{h}(P_f)) & \\ \tilde{h} \nearrow & & \searrow \{h_\tau^{C_j}\} \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_\tau} & (R, h_\tau(P_f)) \end{array}$$

We can now define a complex structure on T^1 component-wise by pulling back complex structures of corresponding components of R by $\{f^{C_i^1}\}$ in the same manner as we did when we defined σ_f on \mathcal{T}_f . For every component C_i^1 we construct a

commutative diagram analogous to the commutative diagram (3.3)

$$\begin{array}{ccc}
 C_i^1 & \xrightarrow{h_1^{C_i^1}} & R_i^1 \\
 f^{C_i^1} \downarrow & & \downarrow f_\tau^{C_i^1} \\
 C_j & \xrightarrow{h_\tau^{C_j}} & R_j
 \end{array} \tag{4.1}$$

Note that all components on the left are topological spheres and all components on the right are Riemann spheres. Thus, the situation is exactly as above with one important exception: $f^{C_i^1}$ is not a self-map but a map between two different topological spheres.

The topological noded surface T^1 is now endowed with complex structure and becomes a Riemann surface with nodes R^1 of the same type as R . Let $h_1: T^1 \rightarrow R^1$ be the map that acts on every component C_i^1 as $h_1^{C_i^1}$. The following diagram summarizes the aforesaid.

$$\begin{array}{ccccc}
 (\mathbb{S}^2, P_f) & \xrightarrow{\tilde{h}_1} & (T^1, \tilde{h}_1(P_f)) & \xrightarrow{h_1} & (R^1, h_1 \circ \tilde{h}_1(P_f)) \\
 \downarrow f & & \downarrow \{f^{C_i^1}\} & & \downarrow \{f_\tau^{C_i^1}\} \\
 & & (T, \tilde{h}(P_f)) & & \\
 & \nearrow \tilde{h} & & \searrow \{h_\tau^{C_j}\} & \\
 (\mathbb{S}^2, P_f) & \xrightarrow{h_\tau} & (R, h_\tau(P_f)) & &
 \end{array} \tag{4.2}$$

We set $\sigma_f(\tau) = \langle \tilde{h}_1 \circ h_1 \rangle$. It is straightforward to check that σ_f is now well-defined as a self-map of $\overline{\mathcal{T}}_f$.

Note that this definition of σ_f on the boundary of the augmented Teichmüller space immediately implies the following.

Proposition 4.1.1. *A stratum \mathcal{S}_Γ is mapped by σ_f into the stratum $\mathcal{S}_{f^{-1}(\Gamma)}$. In particular, σ_f -invariant boundary strata are in one-to-one correspondence with completely invariant multicurves.*

Fix a component $C' := C_i^1$ of the noded surface T^1 and the corresponding cover $f^{C'}$ that sends C' to a component $C := C_j$ of the noded surface T . Select points a, b and c from P_f in such a way that no two points from $\{a, b, c\}$ are separated from C by any single curve from Γ (we say that a, b, c *single out* the component C). Obviously, any triple of points (that are not on nodes) on a noded surface

of genus 0 singles out exactly one of its components. Similarly, choose (possibly different) points a', b' and c' in P_f that single out C' in T^1 . For all points τ in \mathcal{T}_f , we normalize the homeomorphisms h_τ and h_1 in the commutative diagram (3.3) so that $h_\tau(a) = h_1(a') = 0$ and $h_\tau(b) = h_1(b') = 1$ and $h_\tau(c) = h_1(c') = \infty$ (or any other selected values). Since f_τ is defined up to pre- and post-compositions with Moebius transformations, fixing these normalization conditions defines all f_τ uniquely.

Let p^C be the naturally defined projection from \mathbb{S}^2 to C that sends connected components of the complement of C to the nodes that separate these components from C and $P_f^C := p^C(P_f)$ be the set of nodes and marked points on C ; define $p^{C'}$ and $P_f^{C'}$ in the same manner. Then a, b and c single out C if and only if p^C is injective on $\{a, b, c\}$. For any point $\tau \in \mathcal{S}_\Gamma$, we define $f_\tau^{C'}$ uniquely by imposing the same normalization on the functions in the commutative diagram (4.1): $h_\tau^C(p^C(a)) = h_1^{C'}(p^{C'}(a')) = 0$ and $h_\tau^C(p^C(b)) = h_1^{C'}(p^{C'}(b')) = 1$ and $h_\tau^C(p^C(c)) = h_1^{C'}(p^{C'}(c')) = \infty$.

Proposition 4.1.2. *Let $\{\tau_n\} \in \mathcal{T}_f$ be a sequence converging to a point $\tau \in \mathcal{S}_\Gamma$. With the normalizations as above $\{f_{\tau_n}\}$ converges uniformly to $f_\tau^{C'}$ on any compact set in the complement of the $h_1^{C'}$ -image of the nodes.*

Proof. Given representing maps h_τ, h_{τ_n} and h_1, h_{1,τ_n} , set $p_n^C = h_\tau^C \circ p^C \circ h_{\tau_n}^{-1}$ and $p_n^{C'} = h_1^{C'} \circ p^{C'} \circ h_{1,\tau_n}^{-1}$ to complete the following commutative cube, where the front side of the cube is the commutative diagram (3.3), the back side of the cube is the commutative diagram (4.1), and the dotted arrows are the corresponding projection maps.

$$\begin{array}{ccccc}
 & & (C', P_f^{C'}) & \xrightarrow{h_1^{C'}} & (\mathbb{P}, h_1^{C'}(P_f^{C'})) \\
 & \nearrow p^{C'} & \downarrow f^{C'} & & \nearrow p_n^{C'} \\
 (\mathbb{S}^2, P_f) & \xrightarrow{h_{1,\tau_n}} & (\mathbb{P}, h_{1,\tau_n}(P_f)) & & \downarrow f_\tau^{C'} \\
 & \downarrow f & \downarrow f_{\tau_n} & & \\
 & & (C, P_f^C) & \xrightarrow{h_\tau^C} & (\mathbb{P}, h_\tau^C(P_f^C)) \\
 & \nearrow p^C & \downarrow f_\tau & & \nearrow p_n^C \\
 (\mathbb{S}^2, P_f) & \xrightarrow{h_{\tau_n}} & (\mathbb{P}, h_{\tau_n}(P_f)) & &
 \end{array} \tag{4.3}$$

For any point $q \in P_f$, we have $h_{\tau_n}(q) \rightarrow h_\tau^C(p^C(q))$. Indeed, for points a, b, c this is by definition, for the rest of the marked points it follows from the fact that the cross ratios of marked points are continuous functions on $\overline{\mathcal{T}}_f$. By assumption, $l(\gamma, \tau_n) \rightarrow 0$ for all $\gamma \in \Gamma$, so with the chosen normalizations, the hyperbolic geodesics

γ_{τ_n} on τ_n are contained in arbitrarily small spherical neighborhoods of the h_τ^C image of the corresponding nodes as n goes to infinity. Thus, since h_{τ_n} and h_τ are defined up to homotopy relative P_f , we can assume that p_n^C is conformal onto $\mathbb{P} \setminus U$ where U is a small neighborhood of $h_\tau^C(P_f^C)$. Following the commutative diagram (4.3), we see that $p_n^{C'}$ is conformal onto $\mathbb{P} \setminus (f_\tau^{C'})^{-1}(U)$.

Proposition 3.4.2 implies that both $\{p_n^C\}$ and $\{p_n^{C'}\}$ uniformly converge to the identity which gives us the desired result. \square

Remark 4.1.3. Note that if we normalize our sequence in the same way as above, but choosing $\{a, b, c\}$ and $\{a', b', c'\}$ so that these triples do not satisfy the conditions described above (which means that $\{a', b', c'\}$ singles out a component C' of $\sigma_f(\tau)$ that does not map to the component C of τ singled out by $\{a, b, c\}$), then f_{τ_n} converges uniformly on any compact set in the complement of the image of the nodes to a constant map. Indeed, with the chosen parametrization the component C' becomes larger and larger as n grows and is mapped into the complement of the component C and this complement becomes smaller and smaller.

The last proposition is not only a useful tool for proving the next theorem but is also of interest in its own right. We see that under the normalization assumptions given above, as τ tends to a boundary point τ_0 in $\overline{\mathcal{T}}_F$, the map f_τ deforms in a continuous fashion with respect to the locally uniform convergence in the complement of the nodes. All possible limits are either constant maps or rational maps of possibly smaller degree that map components of $\sigma_f(\tau_0)$ to the corresponding components of τ_0 .

Theorem 4.1.4. *The map σ_f as defined above is continuous on $\overline{\mathcal{T}}_f$.*

Proof. We see that this map by definition preserves the product structure on every stratum in the following sense. If $\mathcal{S}_\Gamma \cong \mathcal{T}_1 \times \mathcal{T}_2 \times \dots \times \mathcal{T}_n$ then for each point $\tau = (\tau^1, \tau^2, \dots, \tau^n) \in \mathcal{S}_\Gamma$ we have

$$\sigma_f(\tau) = (\sigma_1(\tau^{i_1}), \sigma_2(\tau^{i_2}), \dots, \sigma_m(\tau^{i_m})) \in \mathcal{S}_{f^{-1}(\Gamma)}$$

where $\sigma_1, \dots, \sigma_m$ are pullback maps for the covers f_1, \dots, f_m from components of τ_1 to components of τ . It follows immediately that σ_f is continuous on each stratum of \mathcal{T}_f . Since every stratum lies on the boundary of finitely many strata, it suffices to show sequential continuity for sequences that lie within a single stratum converging to a boundary point. Moreover, we can assume that the stratum is \mathcal{T}_f itself; the other cases follow if we apply the same argument to $\sigma_1, \dots, \sigma_m$.

We are going to show now that for any $\{\tau_n\} \in \mathcal{T}_f$ such that $\tau_n \rightarrow \tau \in \partial\mathcal{T}_f$, we have $l(\gamma', \sigma_f(\tau_n)) \rightarrow l(\gamma', \sigma_f(\tau))$ for every γ' , which will conclude the proof of the theorem by Proposition 3.3.3. Denote $\sigma_f(\tau_n) = \tau'_n$ and $\sigma_f(\tau) = \tau'$ to simplify the notation. We consider three cases: when γ' is a node of τ' , when γ' intersects at least one of the nodes, and the rest of the homotopy classes of simple closed curves.

First, let us look at the case when γ' is a node of τ' (i.e. $l(\gamma', \tau') = 0$). Then, by definition, γ' is homotopic to at least one preimage of a curve γ which is a node of τ . Then f maps this preimage onto γ as a cover of some certain degree, say d . The corresponding preimage δ of the geodesic homotopic to γ in τ_n has length equal to $d \cdot l(\gamma, \tau_n)$ with respect to the Poincaré metric on $\mathbb{P} \setminus f_{\tau_n}^{-1}(P_f)$. Since filling in some of the punctures decreases Poincaré metric we get that $l(\gamma', \tau'_n) \leq l(\delta, \tau'_n) \leq d l(\gamma, \tau_n)$. Since $l(\gamma, \tau_n) \rightarrow l(\gamma, \tau) = 0$ we conclude that $l(\gamma', \tau'_n) \rightarrow 0 = l(\gamma', \tau')$.

The second case is when $l(\gamma', \tau') = \infty$. In this case γ' must have positive intersection number with at least one node δ of τ' . We already know that $l(\delta, \tau'_n)$ tends to 0. By the Collaring Lemma, it follows that $l(\gamma', \tau'_n)$ tends to infinity.

We conclude the proof by showing that $l(\gamma', \tau'_n) \rightarrow l(\gamma', \tau')$ for the rest of the curves (i.e., when $l(\gamma', \tau') \notin \{0, \infty\}$). Let C' be the component of τ' that contains γ' and C be the corresponding component of τ . Fix normalization conditions for representing homeomorphisms h_τ, h_1 and $h_{\tau_n}, h_{1, \tau_n}$ as above so that Proposition 4.1.2 applies.

Denote by $P_{f,n}$ the h_{1, τ_n} -image of P_f ; let ρ_n be the hyperbolic distance element on $\mathbb{P} \setminus P_{f,n}$ and ρ be the hyperbolic distance element on $\mathbb{P} \setminus h_1^{C'}(P_f^{C'})$. Define ρ_n^1 to be the hyperbolic distance element on $\mathbb{P} \setminus U_n$, where U_n is a small neighborhood of $h_1^{C'}(P_f^{C'})$ that contains $P_{f,n}$. Recall that $p_n^{C'} = h_1^{C'} \circ p^C \circ h_{1, \tau_n}^{-1}$ converges uniformly to the identity (see the proof of Proposition 4.1.2) which implies that $\{U_n\}$ can be chosen so that the intersection thereof is $h_1^{C'}(P_f^{C'})$. Define ρ_n^2 to be the hyperbolic distance element on $\mathbb{P} \setminus P'_{f,n}$ where $P'_{f,n} \subset P_{f,n}$ is such that $P'_{f,n}$ has exactly one point in every connected component of U_n . Then, by the Schwarz lemma, we have $\rho_n^1 \geq \rho_n \geq \rho_n^2$.

Applying Proposition 3.4.1, we get that point-wise $\rho_n^1 \rightarrow \rho$. On the other hand, we clearly have $\rho_n^2 \rightarrow \rho$ because $\{(\mathbb{P}, P'_{f,n})\}$ is a converging sequence in the Teichmüller space of $(\mathbb{S}^2, P_f^{C'})$. This shows that $\rho_n \rightarrow \rho$ point-wise.

Note that geodesics on all $\mathbb{P} \setminus P_{f,n}$ in the same homotopy class as γ' live in a compact subset of $\mathbb{P} \setminus h_1^{C'}(P_f^{C'})$ by the Collaring Lemma. But on any compact set, the convergence of the hyperbolic length elements will be uniform. One easily deduces

$l(\gamma', \tau'_n) \rightarrow l(\gamma', \tau')$. Since γ' was chosen arbitrarily, we conclude that $l(\gamma', \tau'_n) \rightarrow l(\gamma', \tau')$ for all homotopy classes γ' of simple closed curves in (\mathbb{S}^2, P_f) . \square

4.2 Classification of invariant boundary strata

As was mentioned above, every invariant boundary stratum corresponds to a completely invariant multicurve Γ . We want to classify the topological behavior of σ_f near invariant boundary strata \mathcal{S}_Γ according to the value of λ_Γ . An invariant stratum \mathcal{S} of $\overline{\mathcal{T}}_f$ will be called *weakly attracting* if there exists a nested decreasing sequence of neighborhoods U_n such that $\sigma_f(U_n) \subset U_n$ and $\bigcap U_n = \mathcal{S}$. An invariant stratum \mathcal{S} of $\overline{\mathcal{T}}_f$ will be called *weakly repelling* if for any compact set $K \subset \mathcal{S}$ there exists a neighborhood $U \supset K$ such that every point of $U \cap \mathcal{T}_f$ escapes from U after finitely many iterations (for every $\tau \in U \cap \mathcal{T}_f$, there exists an $n \in \mathbb{N}$ such that $\sigma_f^n(\tau) \notin U$).

Proposition 4.2.1. *If $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_m\}$ is a simple obstruction, then \mathcal{S}_Γ is weakly attracting.*

Proof. By Proposition 3.1.1, we can choose a vector $v > 0$ such that $M_\Gamma v \geq v$. Consider $V_n \subset \overline{\mathcal{T}}_f$, the set of all points $\tau = \langle h \rangle$ of the augmented Teichmüller space for which there exist mutually disjoint annuli A_i homotopic to γ_i in (\mathbb{S}^2, P_f) such that $\text{mod } h(A_i) > nv_i$ for $i = \overline{1, m}$.

Construct disjoint annuli B_i for $i = \overline{1, m}$ in (\mathbb{S}^2, P_f) that contain the union of all components $A_{i,j,k}$ of the preimage of A_j that are homotopic to γ_i . Pick a diffeomorphism h_1 that represents $\sigma_f(\tau)$ (i.e. $\sigma_f(\tau) = \langle h_1 \rangle$). By definition, $f_\tau = h \circ f \circ h^{-1}$ is holomorphic and non-ramified on all $h_1(A_{i,j,k})$. Therefore,

$$\text{mod } h_1(A_{i,j,k}) = (\deg f|_{A_{i,j,k}} : A_{i,j,k} \rightarrow A_j)^{-1} \text{mod } h(A_j).$$

It follows from the Grötzsch inequality (see Theorem 2.4.1) that

$$\text{mod } h_1(B_i) \geq \sum_{A_{i,j,k}} \text{mod } h_1(A_{i,j,k}) = \sum_{A_{i,j,k}} (\deg f|_{A_{i,j,k}} : A_{i,j,k} \rightarrow A_j)^{-1} \text{mod } h(A_j).$$

If we write this inequality in vector form we get simply

$$\text{mod } h_1(B_i) \geq M_\Gamma \text{mod } h(A_i) > M_\Gamma(nv) \geq nv.$$

Thus $\sigma_f(\tau) \in V_n$ for all $\tau \in V_n$.

As mentioned above, large annuli in homotopy classes of γ_i exist if and only if the

lengths $l(\gamma_i, \tau)$ are short. It follows that $\bigcap V_n = \overline{\mathcal{S}_\Gamma}$. Denote by U_n the intersections of V_n with the union of all strata $\mathcal{S}_{\hat{\Gamma}}$ where $\hat{\Gamma} \subseteq \Gamma$. Then clearly $\bigcap U_n = \mathcal{S}_\Gamma$, and U_n are still invariant since $\hat{\Gamma} \subseteq \Gamma$ implies $f^{-1}(\hat{\Gamma}) \subseteq f^{-1}(\Gamma) = \Gamma$. \square

We will need the following (see Proposition 1.2.2 in [KH95])

Proposition 4.2.2. *Let $M \in \mathbb{C}^{m \times m}$ be a matrix such that all eigenvalues of M have absolute value strictly less than δ . Then there exists a norm $\|\cdot\|$ on \mathbb{C}^m such that the operator norm satisfies $\|M\| \leq \delta$.*

Proof. Clearly, it is sufficient to construct such a norm for any matrix in the conjugacy class of M , hence, we assume that M is in the Jordan normal form. Moreover, we can further assume that M is a Jordan block since if M consists of several Jordan blocks, we can construct the desired norm by taking, for example, the sum of norms defined in the same way as below on subspaces corresponding to each block.

For the Jordan block

$$M = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

we define $\|x\| = \max_{i=1, \dots, m} \{\varepsilon^{m-i} |x_i|\}$, where $\varepsilon = \delta - |\lambda|$ and $x = (x_1, x_2, \dots, x_m)^T$. We claim that this norm satisfies $\|Mx\| \leq \delta \|x\|$. Indeed, suppose on the contrary that $\|Mx\| > \delta \|x\|$. Take the maximal number from $\{\varepsilon^{m-1} |\lambda x_1 + x_2|, \varepsilon^{m-2} |\lambda x_2 + x_3|, \dots, \varepsilon |\lambda x_{m-1} + x_m|, |\lambda x_m|\}^T$. It can not be the m -th one, since then we would have $\|Mx\| = |\lambda x_m| < \delta |x_m| < \delta \max_{i=1, \dots, m} \{\varepsilon^{m-i} |x_i|\} = \delta \|x\|$. But then $\|Mx\| = \varepsilon^{m-i} |\lambda x_i + x_{i+1}|$ must satisfy both:

$$\begin{cases} \varepsilon^{m-i} (|\lambda| |x_i| + |x_{i+1}|) \geq \varepsilon^{m-i} |\lambda x_i + x_{i+1}| > (|\lambda| + \varepsilon) \varepsilon^{m-i} |x_i|, \\ \varepsilon^{m-i} (|\lambda| |x_i| + |x_{i+1}|) \geq \varepsilon^{m-i} |\lambda x_i + x_{i+1}| > (|\lambda| + \varepsilon) \varepsilon^{m-i-1} |x_{i+1}|, \end{cases}$$

which reduces to a contradiction

$$\begin{cases} |x_{i+1}| > \varepsilon |x_i|, \\ \varepsilon |\lambda| |x_i| > |\lambda| |x_{i+1}|. \end{cases}$$

\square

Recall the following analytic tool from [DH93, Theorem 7.1].

Proposition 4.2.3. *Let X be a Riemann surface and $P \subset X$ a finite set. Set $X' = X \setminus P$, $p = \#P$, and choose $L < 2\log(\sqrt{2} + 1)$. Let γ be a simple closed geodesic on X , and $\{\gamma'_1, \dots, \gamma'_s\}$ be the closed geodesics of X' homotopic to γ in X and of length $< L$. Then*

$$1/l - 2/\pi - (p+1)/L < \sum_{i=1}^s 1/l'_i < 1/l + 2(p+1)/\pi.$$

Define $Z(\Gamma, \tau) = (1/l(\tau, \gamma_1), \dots, 1/l(\tau, \gamma_m))^T$. Then $\|Z(\Gamma, \tau)\|$ can be roughly thought as the inverse of distance between τ and the boundary of $\overline{\mathcal{T}}_f$, i.e. the larger $\|Z(\Gamma, \tau)\|$ is, the closer τ is to some stratum $\mathcal{S}_{\hat{\Gamma}}$ with $\hat{\Gamma} \subseteq \Gamma$.

Proposition 4.2.4. *Let $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_m\}$ be an invariant multicurve with $\lambda_\Gamma < 1$. Pick a norm $\|\cdot\|$ for M_Γ on \mathbb{R}^m as in Proposition 4.2.2 with $\lambda_\Gamma < \delta < 1$. Take*

$$U(\Gamma) = \left\{ \tau \in \mathcal{T}_f \mid \inf_{\gamma \notin \Gamma} l(\gamma, \tau) \geq L \right\},$$

where $0 < L < 2\log(\sqrt{2} + 1)$. Then for every $\delta' > \delta$ there exists $T(L, \delta, \delta') > 0$ such that $\|Z(\Gamma, \sigma_f(\tau))\| < \delta'\|Z(\Gamma, \tau)\|$ for all $\tau \in U(\Gamma)$ with $\|Z(\Gamma, \tau)\| > T(L, \delta, \delta')$.

Proof. Let τ and $\tau_1 = \sigma_f(\tau)$ be represented by h and h_1 respectively. Let us apply Proposition 4.2.3 to the surface $X = \mathbb{P} \setminus h_1(P_f)$ and its finite subset $P = h_1(f^{-1}(P_f) \setminus P_f)$. Every geodesic on $X \setminus P$ is mapped by a non-ramified cover f_τ onto a geodesic of $\mathbb{P} \setminus h(P_f)$. Therefore, those geodesics on $X \setminus P$ that are not preimages of geodesics with homotopy classes in Γ have lengths at least L . Denote as before by $\gamma_{i,j,k}$ preimages of γ_j that are homotopic to γ_i for all pairs i, j . Then $l(\gamma_{i,j,k}, X) = (\deg f|_{\gamma_{i,j,k}} : \gamma_{i,j,k} \rightarrow \gamma_j)l(\gamma_j, \tau)$. Also note that $\#(f^{-1}(P_f) \setminus P_f) \leq d_f \#P_f = d_f p_f$. Thus, we get for all $i = \overline{1, m}$:

$$1/(l(\gamma_i, \tau_1)) < 2/\pi + (dp_f + 1)/L + \sum_{\gamma_{i,j,k}} (\deg f|_{\gamma_{i,j,k}} : \gamma_{i,j,k} \rightarrow \gamma_j)^{-1} l(\gamma_j, \tau)^{-1}.$$

Expressing these inequalities in vector form, we get $Z(\Gamma, \tau_1) < M_\Gamma Z(\Gamma, \tau) + c$, where c is a constant vector depending only on L with $\|c\| = C(L)$. By Proposition 4.2.2 we get:

$$\|Z(\Gamma, \tau_1)\| < \|M_\Gamma Z(\Gamma, \tau) + c\| \leq \|M_\Gamma\| \|Z(\Gamma, \tau)\| + \|c\| < \delta \|Z(\Gamma, \tau)\| + C(L).$$

It is evidently sufficient to take $T(L, \delta, \delta') = C(L)/(\delta' - \delta)$. \square

We can extend the setting of the previous proposition by considering any com-

pletely invariant multicurve Γ which is not a simple obstruction. We write

$$M_\Gamma = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix},$$

where the leading eigenvalue λ_1 of M_{11} is less than 1. Denote by s the dimensions of M_{11} . As before, using Proposition 4.2.2, define a norm on \mathbb{R}^s such that $\|M_{11}\| \leq \delta < \lambda_1$. Extend this norm to a semi-norm on \mathbb{R}^m which depends only on the first s coordinates. It is straightforward to check that the proof of Proposition 4.2.4 works in this setting. We conclude with the following proposition.

Proposition 4.2.5. *Let $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_m\}$ be a completely invariant multicurve which is not a simple obstruction. Fix $L \in \mathbb{R}$ such that $0 < L < 2 \log(\sqrt{2} + 1)$ and define*

$$U(\Gamma) = \left\{ \tau \in \mathcal{T}_f \mid \inf_{\gamma \notin \Gamma} l(\gamma, \tau) \geq L \right\}.$$

Then there exists a semi-norm $\|\cdot\|$ on \mathbb{R}^m and two real numbers $T(L) > 0$ and $0 < \delta < 1$ such that $\|Z(\Gamma, \sigma_f(\tau))\| < \delta \|Z(\Gamma, \tau)\|$ for all $\tau \in U(\Gamma)$ with $\|Z(\Gamma, \tau)\| > T(L)$.

Corollary 4.2.6. *If $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_m\}$ is a completely invariant multicurve which is not a simple obstruction, then \mathcal{S}_Γ is weakly repelling.*

Proof. For any compact $K \subset \mathcal{S}_\Gamma$ we have $\inf_{\tau \in K, \gamma \notin \Gamma} l(\gamma, \tau) = k > 0$. Choose $L = (1/2) \min\{k, 2 \log(\sqrt{2} + 1)\}$. Consider

$$U = \left\{ \tau \in \mathcal{T}_f \mid \inf_{\gamma \notin \Gamma} l(\gamma, \tau) \geq L \text{ and } \|Z(\Gamma, \tau)\| > T \right\},$$

where $T = T(L)$ is as in Proposition 4.2.5. Clearly $\bar{U} \supset K$. Suppose that there exists a point $\tau \in \mathcal{T}_f$ that does not escape from U , i.e., $\sigma_f^n(\tau) \in U$ for all n . Then, on one hand, we have $\|Z(\Gamma, \sigma_f^n(\tau))\| > T$ for all n and, on the other hand, $\|Z(\Gamma, \sigma_f^{n+1}(\tau))\| < \delta \|Z(\Gamma, \sigma_f^n(\tau))\|$ which is a contradiction. \square

4.3 Proofs of Thurston's and Canonical Obstruction Theorems

In the previous section, we described the behavior of σ_f near invariant boundary strata. The understanding of the action of σ_f near infinity plays a key role in our proof of Thurston's theorem.

The following proposition is essentially [DH93, Lemma 5.2] (property *iii.* is not stated in [DH93], but follows from the proof given there).

Proposition 4.3.1. *There exists an intermediate cover \mathcal{M}'_f of \mathcal{M}_f (so that $\mathcal{T}_f \xrightarrow{\pi_1} \mathcal{M}'_f \xrightarrow{\pi_2} \mathcal{M}_f$ are covers and $\pi_2 \circ \pi_1 = \pi$) such that*

- i. π_2 is finite,*
- ii. the diagram*

$$\begin{array}{ccc}
 \mathcal{T}_f & \xrightarrow{\sigma_f} & \mathcal{T}_f \\
 \pi \downarrow & \searrow \pi_1 & \downarrow \pi \\
 & \mathcal{M}'_f & \\
 \pi \downarrow & \swarrow \pi_2 & \searrow \tilde{\sigma}_f \\
 \mathcal{M}_f & & \mathcal{M}_f
 \end{array} \tag{4.4}$$

commutes for some map $\tilde{\sigma}_f: \mathcal{M}'_f \rightarrow \mathcal{M}_f$,

- iii. If $\pi_1(\tau_1) = \pi_1(\tau_2)$ then $f_{\tau_1} = f_{\tau_2}$ up to pre- and post-composition by Moebius transformations.*

In particular, for every $m \in \mathcal{M}_f$ there are only finitely many different f_τ when $\tau \in \pi^{-1}(m)$.

Remark 4.3.2. Note that \mathcal{M}'_f is a quotient of \mathcal{T}_f by a subgroup G of the mapping class group of finite index. Then the quotient $\overline{\mathcal{M}'_f}$ of $\overline{\mathcal{T}_f}$ by the same subgroup will be a compactification of \mathcal{M}'_f . The covers π_1 and π_2 can be extended to the corresponding augmented spaces so that $\pi_2 \circ \pi_1 = \pi$ still holds. As in the case of the compactified moduli space, we parametrize boundary strata by equivalence classes of multicurves, two classes of simple closed curves being equivalent if one can be mapped to the other by the action of G . Evidently, the whole diagram above also extends to the augmented spaces.

Since the mapping class group acts by isometries with respect to both the Teichmüller and Weil-Petersson metrics, both metrics can be projected to \mathcal{M}_f and \mathcal{M}'_f .

Proof of Theorem 3.1.2 (Thurston's Theorem). Necessity of criterion. Suppose Γ is a Thurston obstruction. By Proposition 3.2.2, it is enough to show that σ_f has no fixed points in \mathcal{T}_f . Suppose, by contradiction, that τ is a fixed point, we know then that every forward orbit must converge to τ (see Section 3.2). We may assume that Γ is simple. By Proposition 4.2.1 the stratum \mathcal{S}_Γ is weakly attracting, so we can choose an invariant neighborhood U of \mathcal{S}_Γ that does not contain τ . But

then orbits of points from U are contained in U and can not converge to τ , which is a contradiction.

Sufficiency of criterion. Pick a point $\tau_0 \in \mathcal{T}_f$ and set $\tau_n = \sigma_f^n(\tau)$. Consider the projection of $\{\tau_n\}$ to \mathcal{M}'_f : $m'_n = \pi_1(\tau_n)$. Let $D = d_T(\tau_0, \tau_1)$ be the Teichmüller distance between τ_0 and τ_1 . Since σ_f is weakly contracting with respect to the Teichmüller metric, $d_T(\tau_i, \tau_{i+1}) \leq D$ for all i . Let m' be an accumulation point of $\{m'_n\}$ in $\overline{\mathcal{M}'_f}$ that belongs to a stratum of minimal possible dimension.

If $m' \in \mathcal{M}'_f$, then $\{\tau_n\}$ converges in \mathcal{T}_f . Indeed, by part *iii* of Proposition 4.3.1, the norm of the coderivative $(d\sigma_f)^* = (f_\tau)_*$ depends only on $\pi_1(\tau)$. We know that $\|(d\sigma_f^k)^*\|_T < 1$, for some $k \in \mathbb{N}$, because f is a Thurston map with hyperbolic orbifold. For the sake of simplicity, we assume that $\|(d\sigma_f)^*\|_T < 1$ (if not, we apply the same argument for σ_f^k). On the π_1 -preimage of any compact subset K of \mathcal{M}'_f , we then have $\sup_{\tau \in \pi^{-1}(K)} \|(d\sigma_f)^*\|_T < 1$. It follows that the Teichmüller distance between τ_n and τ_{n+1} is contracted by some definite factor $\lambda < 1$ by σ_f (i.e. $d_T(\tau_{n+1}, \tau_{n+2}) \leq \lambda d_T(\tau_n, \tau_{n+1})$) when $\pi_1(\tau_n) \in K$. We take K to be a closed ball of radius r around m' in \mathcal{M}'_f . Then infinitely many τ_n are in K , thus $d_T(\tau_n, \tau_{n+1})$ tends to 0.

Take N such that the distance between τ_N and τ_{N+1} is smaller than $r(1 - \lambda)/2$ and the distance between m'_N and m' is less than $r/2$. Since $m'_N \in K$ we get $d_T(\tau_{N+1}, \tau_{N+2}) \leq \lambda d_T(\tau_N, \tau_{N+1}) < \lambda r(1 - \lambda)/2$. Since $d_T(\tau_N, \tau_{N+1}) \sum_i \lambda^i < r/2$, we see by induction that $m'_{N+k} \in K$ and $d_T(\tau_{N+k}, \tau_{N+k+1}) \leq \lambda^k d_T(\tau_N, \tau_{N+1})$ for all $k \in \mathbb{N}$. Thus $\{\tau_n\}$ converges at least as fast as a geometric series to a fixed point of σ_f . In this case, f is Thurston equivalent to a rational function.

From now on we assume that $m' \in \mathcal{S}_{[\Gamma']}$ with $\Gamma' = \{\gamma'_1, \dots, \gamma'_s\} \neq \emptyset$. Since we chose m' on the stratum of minimal possible dimension, it follows that there exists $L \in (0, 2 \log(\sqrt{2} + 1))$ such that for all τ_n there exist at most s different simple closed geodesics of length less than L . The Collaring Lemma implies that these geodesics are mutually disjoint (see Corollary 2.4.4). Choose $L_1 > 0$ satisfying $L_1 < e^{-D}L/d_f$ and $1/L_1 > e^D(2/\pi + (d_f p_f + 1)/L)$. Consider n such that $l(\gamma_i, \tau_n) < L_1$ with $\gamma_i \in [\gamma'_i]$ for all $i = \overline{1, s}$. We claim that $\Gamma_n = \{\gamma_1, \dots, \gamma_s\}$ is completely invariant.

Indeed, since $\log l(\gamma, \tau)$ is 1-Lipschitz, we have $e^{-D}l(\gamma, \tau_n) \leq l(\gamma, \tau_{n+1}) \leq e^D l(\gamma, \tau_n)$. On the other hand, every essential preimage of any γ_i has length at most $d_f L_1 < e^{-D}L$ so it must be homotopic to a curve in Γ_n ; this proves invariance of Γ_n . If some γ_i were homotopic to no preimage of curves from Γ_n , then by Proposition 4.2.3 we would get $1/l(\gamma_i, \tau_{n+1}) < 2/\pi + (d_f p_f + 1)/L < e^{-D}/L_1 < e^{-D}/l(\gamma_i, \tau_n)$; this proves complete invariance.

Take a subsequence $\{m'_{n_k}\}$ that converges to m' and such that for each n_k there exist $\gamma_i \in [\gamma'_i]$ for $i = \overline{1, s}$ such that $l(\gamma_i, \tau_{n_k}) < L_1$. Define Γ_{n_k} as above for each k . Then for any pair j, k there exists an element g of the deck transformation group G corresponding to the covering π_1 (see Proposition 4.3.1 and Remark 4.3.2) such that $g(\Gamma_{n_k}) = \Gamma_{n_j}$. Consider a pair of points τ and $g(\tau)$; the commutative diagram (4.4) yields

$$\pi(\sigma_f(g(\tau))) = \tilde{\sigma}_f(\pi_1(g(\tau))) = \tilde{\sigma}_f(\pi_1(\tau)) = \pi(\sigma_f(\tau)).$$

Hence, there exists an element h of the pure mapping class group such that $\sigma_f(g(\tau)) = h(\sigma_f(\tau))$. Since both Γ_{n_j} and Γ_{n_k} are completely invariant, we get

$$g(\Gamma_{n_k}) = \Gamma_{n_j} = f^{-1}(\Gamma_{n_j}) = f^{-1}(g(\Gamma_{n_k})) = h(f^{-1}(\Gamma_{n_k})) = h(\Gamma_{n_k}).$$

This implies that $h(\gamma) = g(\gamma)$ for all $\gamma \in \Gamma_{n_k}$. It follows that Γ_{n_k} have the same Thurston matrix M for all k .

By assumption Γ_{n_1} is not an obstruction. Select $T(L)$ and a semi-norm on \mathbb{R}^s as in Proposition 4.2.5. Consider the sets

$$U(\Gamma_{n_k}, T) = \left\{ \tau \in \mathcal{T}_f \mid \inf_{\gamma \notin \Gamma_{n_k}} l(\gamma, \tau) \geq L \text{ and } \|Z(\Gamma_{n_k}, \tau)\| > T \right\},$$

where $T \geq e^D T(L)$ and is large enough so that $\tau_1 \notin U(\Gamma_{n_k}, T)$ for all k (this is possible since for $k = \overline{1, \infty}$ the set $Z(\Gamma_{n_k}, \tau)$ is bounded in \mathbb{R}^s for all $\tau \in \mathcal{T}_f$, because there are only finitely many short curves on the Riemann surface corresponding to τ). Since $\{m'_{n_k}\}$ converges to m' , we can pick the smallest $n = n_k$ such that $\tau_n \in U(\Gamma_n, T)$. We use the Lipschitz condition again to get that $\tau_{n-1} \in U(\Gamma_n, Te^{-D})$. Proposition 4.2.5 yields $\|Z(\Gamma_n, \tau_n)\| < \delta \|Z(\Gamma_n, \tau_{n-1})\|$ because $Te^{-D} \geq T(L)$; in particular, we see that $\tau_{n-1} \in U(\Gamma_n, T)$. Continuing by induction, we see that $\tau_{n_k-1} \in U(\Gamma_n, T)$ which is a contradiction. Therefore, Γ_{n_1} is a simple obstruction and we are done. □

Proof of Theorem 3.2.6.

As above, take a point $\tau_0 \in \mathcal{T}_f$ and set $\tau_n = \sigma_f^n(\tau), m'_n = \pi_1(\tau_n)$.

Pick an accumulation point m' of $\{m'_n\}$ on a stratum $\mathcal{S}_{[\Gamma']}$ of \mathcal{M}'_f of smallest dimension possible. If Γ' is empty, it follows that all m'_n lie in a compact subset of \mathcal{M}'_f and hence $l(\gamma, \tau_n) > L$ for all $n = \overline{1, \infty}$ and γ . Then the canonical obstruction must be empty and the statement of the theorem follows.

Suppose now that Γ' is not empty. As we have shown in the previous proof, if τ_n is close enough to the stratum \mathcal{S}_Γ of \mathcal{T}_f where $\Gamma \in [\Gamma']$, then Γ is a simple

obstruction. Proposition 4.2.1 tells us that once τ_n is in a small neighborhood of \mathcal{S}_Γ then $\{\tau_m\}$ stays in that neighborhood for all $m > n$. Therefore, the accumulation set of $\{m'_n\}$ must be a subset of $\overline{\mathcal{S}_{[\Gamma]}}$, moreover $l(\gamma, \tau_n) \rightarrow 0$ for all $\gamma \in \Gamma$. As $\mathcal{S}_{[\Gamma]}$ was a stratum of minimal dimensions to contain an accumulation point of m'_n , it follows that the accumulation set of $\{m'_n\}$ lies in a compact subset of $\mathcal{S}_{[\Gamma]}$, and thus $l(\gamma, \tau_n) > L$ for all $n = \overline{1, \infty}$ and $\gamma \notin \Gamma$ with some constant $L > 0$. This shows that Γ is the canonical obstruction for f and proves the statement of the theorem. \square

Proof of Theorems 3.2.3 and 3.2.4.

As we have shown in the proof of Thurston's theorem, if all m'_n lie in a compact subset of \mathcal{M}'_f then f has a fixed point. It follows that f is equivalent to a rational map if and only if the canonical obstruction Γ_f is empty. The statement of Theorem 3.2.4 follows immediately from the previous proof. \square

From the previous arguments, we actually get a slightly stronger statement.

Corollary 4.3.3. *If Γ_f is not empty then it is a simple completely invariant Thurston obstruction.*

The following proposition shows the relation between simple Thurston obstructions and invariant multicurves.

Proposition 4.3.4. *For any simple Thurston obstruction Γ there exists an invariant multicurve Γ' such that $\Gamma \subset \Gamma'$.*

Proof. Take a vector v such that $v > 0$ and $M_\Gamma v \geq v$ (this is possible by Proposition 3.1.1), and each component of v is larger than $d_f \pi / (\ln(3 + 2\sqrt{2}))$. Pick a point $\tau \in \mathcal{T}_f$ so that there exist disjoint annuli A_i on the Riemann surface corresponding to τ such that for each i , the annulus A_i has modulus v_i and is homotopic to $\gamma_i \in \Gamma$. We define annulus B_j on the Riemann surface corresponding to $\sigma_f(\tau)$ to be the minimal annulus containing all f_τ -preimages of annuli A_i that are homotopic to γ_j . Applying the Grötzsch inequality (Theorem 2.4.1) the same way as in the proof of Proposition 4.2.1, we see that for each i , the annulus B_i has modulus at least v_i .

Let γ' be an essential preimage component of some $\gamma_i \in \Gamma$. The corresponding f_τ -preimage of A_i has modulus at least $v_i/d_f \geq \pi / (\ln(3 + 2\sqrt{2}))$ because it is mapped by f_τ as an unbranched cover of degree at most d_f . Corollary 2.4.5 implies that γ' has intersection number zero with all curves in Γ . Since γ' was chosen arbitrarily, we infer that $\Gamma_1 = \Gamma \cup f^{-1}(\Gamma)$ is a multicurve. It is obvious that Γ_1 is also a simple obstruction. By induction, we construct simple obstructions $\Gamma_{k+1} = \Gamma_k \cup f^{-1}(\Gamma_k)$

for all $k \in \mathbb{N}$. Since the sequence Γ_k is increasing and each of Γ_k can have at most $p_f - 3$ elements, for some k , we have $\Gamma_{k+1} = \Gamma_k$ and, hence, $f^{-1}(\Gamma_k) \subset \Gamma_k$. \square

4.4 The Thurston boundary of the Teichmüller space

For a more detailed introduction to the notion of the Thurston boundary we address the reader to [IT92]. Let S be the set of all free homotopy classes of simple closed curves on $\mathbb{S}^2 \setminus P_f$. Then the function $l(\gamma, \tau): S \times \mathcal{T}_f \rightarrow \mathbb{R}^+$ can be viewed as a map from \mathcal{T}_f to $\mathbb{R}\mathbb{P}^S$. This map can be proven to be an analytic injection (cf. Section 3.3) with the image homeomorphic to a ball of the same dimension as \mathcal{T}_f . The boundary of this ball in $\mathbb{R}\mathbb{P}^S$ is called the Thurston boundary of T_f . Points of the Thurston boundary are represented by positive real-valued functions on S where two functions correspond to the same point if and only if their ratio is constant. For example, for any $\gamma \in S$ the topological intersection number $\langle \gamma, \cdot \rangle$ is a function on S corresponding to a point on the Thurston boundary and the set of all such points is dense. The Thurston boundary can be identified with the set \mathcal{PMF} of projective measured foliations [Thu88]. We will use the following basic fact.

Proposition 4.4.1. *Suppose that for a sequence $\{\tau_n\} \in \mathcal{T}_f$, the lengths $\{l(\gamma, \tau_n)\} \rightarrow 0$ and $l(\delta, \tau_n) > \varepsilon$ for all $n \in \mathbb{N}$ and all δ that are not homotopic to γ , with some $\varepsilon > 0$. Then $\{\tau_n\}$ converges to the point $\langle \gamma, \cdot \rangle$ in the Thurston boundary.*

Thurston's pullback map σ_f can be decomposed into a composition of two maps as follows. Suppose $g: R_1 \rightarrow R_2$ is a covering map between two surfaces R_1 and R_2 of finite type. Then one can define the usual pullback map $g^*: \mathcal{T}(R_2) \rightarrow \mathcal{T}(R_1)$. If $i: R_1 \rightarrow R_2$ is an inclusion map between two surfaces R_1 and R_2 of finite type (that is a map that fills in some of the punctures of R_1) then one can define the push-forward map $i_*: \mathcal{T}(R_1) \rightarrow \mathcal{T}(R_2)$ (also called the forgetful map) that just forgets the information about the erased punctures. In our setting, we have $\sigma_f = i_* \circ g^*$ where $g = f|_{\mathbb{S}^2 \setminus f^{-1}(P_f)}$ and $i = \text{id}|_{\mathbb{S}^2 \setminus f^{-1}(P_f)}$.

It is evident that the action of $g^*: \mathcal{T}_f \rightarrow \mathcal{T}(\mathbb{S}^2, f^{-1}(P_f))$ could be continuously extended to the Thurston boundary using the natural action of g^* on measured foliations. However, there is no natural way to push measured foliations forward.

Proposition 4.4.2. *If $p_f > 3$ then i_* has no continuous extension to the Thurston boundary of $\mathcal{T}(\mathbb{S}^2, f^{-1}(P_f))$.*

Proof. Take a simple closed curve γ in $\mathbb{S}^2 \setminus f^{-1}(P_f)$ that separates two points A and B of $f^{-1}(P_f)$ such that $A \in P_f$ and $B \notin P_f$ from all other points of $f^{-1}(P_f)$. Connect

A and B by a simple path δ that does not intersect γ . Fix a complex structure τ on $\mathbb{S}^2 \setminus f^{-1}(P_f)$ and start moving A towards B along δ . The obtained path δ_1 in $\mathcal{T}(\mathbb{S}^2, f^{-1}(P_f))$ tends to the point on the Thurston boundary defined by $\langle \gamma, \cdot \rangle$ by Proposition 4.4.1. Indeed, the length of any curve α that has zero intersection number with γ is bounded below by the length of α on the Riemann surface obtained from τ by filling in the puncture at A , and the length of γ clearly tends to 0.

If we depart from the same initial complex structure and start moving B to A along the path δ , we get a new path δ_2 in $\mathcal{T}(\mathbb{S}^2, f^{-1}(P_f))$ with the same limit. It is clear that the limits of $i_*(\delta_1)$ and $i_*(\delta_2)$ are different in \mathcal{T}_f . \square

The previous proposition is the moral reason why σ_f can not be extended to the Thurston boundary. But since the image of g^* is by far not the whole of $\mathcal{T}(\mathbb{S}^2, f^{-1}(P_f))$ (it can not have dimension greater than the dimension of \mathcal{T}_f), we have to say a little bit more. If we assume that σ_f extends continuously to the Thurston boundary, we get the following necessary condition on f .

Proposition 4.4.3. *Suppose that σ_f extends continuously to the Thurston boundary. If for some essential simple closed curve γ in $\mathbb{S}^2 \setminus P_f$, all f -preimages of γ are non-essential, then σ_f is constant on the stratum $\mathcal{S}_{\{\gamma\}}$.*

Proof. By Proposition 4.4.1, any sequence $\{\tau_n\} \in \mathcal{T}_f$ that converges to a point on $\mathcal{S}_{\{\gamma\}}$ in the augmented Teichmüller space also converges to $\langle \gamma, \cdot \rangle$ in the Thurston compactification. Since the stratum $\mathcal{S}_{\{\gamma\}}$ is mapped into \mathcal{T}_f by Proposition 4.1.1, we see that σ_f must be constant on it. Indeed, for any $\tau \in \mathcal{S}_{\{\gamma\}}$, we can consider a sequence $\{\tau_n\} \in \mathcal{T}_f$ converging to τ . We get $\sigma_f(\tau) = \lim \sigma_f(\tau_n) = \sigma_f(\langle \gamma, \cdot \rangle)$. \square

We conclude by constructing explicit examples where the previous condition is violated.

Theorem 4.4.4. *There exist Thurston maps f such that Thurston's pullback map does not extend to the Thurston boundary of the Teichmüller space.*

Proof. We start with a Thurston map (f, P_f) which is a topological polynomial (i.e. there exists a fixed point $\infty \in P_f$ that has no f -preimages other than itself) with non-constant σ_f . We may assume that f has two fixed points A and B outside of P_f because otherwise we can create extra fixed points by applying a homotopy relative to P_f . Consider the Thurston map $f' = (f, P_f \cup \{A, B\})$.

Let γ be a simple closed curve that separates points A and B from P_f . A component δ of the f -preimage of γ is essential if and only if it also separates A and

B from P_f because the complementary component of δ that does not contain ∞ can contain no marked points except A and B . Thus, if we assume that all curves that separate points A and B from P_f have essential preimages, we easily get that f is a homeomorphism. We fix such a γ that has no essential preimages.

Denote by $P_A : \mathcal{T}(\mathbb{S}^2, P_f \cup \{A, B\}) \rightarrow \mathcal{T}(\mathbb{S}^2, P_f \cup \{B\})$ and $P_{AB} : \mathcal{T}(\mathbb{S}^2, P_f \cup \{A, B\}) \rightarrow \mathcal{T}(\mathbb{S}^2, P_f)$ the canonical projections between the respective Teichmüller spaces. Note that $\mathcal{S}_{\{\gamma\}}$ is canonically isomorphic to $\mathcal{T}(\mathbb{S}^2, P_f \cup \{B\})$. We get the following commutative diagram.

$$\begin{array}{ccc}
 \mathcal{S}_{\{\gamma\}} = \mathcal{T}(\mathbb{S}^2, P_f \cup \{B\}) & \xrightarrow{\sigma_{f'}} & \mathcal{T}(\mathbb{S}^2, P_f \cup \{A, B\}) \\
 \downarrow P_A & & \downarrow P_{AB} \\
 \mathcal{T}(\mathbb{S}^2, P_f) & \xrightarrow{\sigma_f} & \mathcal{T}(\mathbb{S}^2, P_f)
 \end{array}$$

Since σ_f is non-constant by the assumption and P_A is surjective, the map $\sigma_{f'}$ is non-constant on $\mathcal{S}_{\{\gamma\}}$. Proposition 4.4.3 implies that $\sigma_{f'}$ can not be continuously extended to the Thurston boundary of the Teichmüller space. \square

4.5 Pilgrim's conjecture

The geometric description of the extension of σ_f to the boundary of the augmented Teichmüller space allows us to understand fairly well what exactly happens when f is obstructed. In this case, by Theorem 3.2.3 the sequence $\sigma_f^n(\tau)$ tends to \mathcal{S}_{Γ_f} for all $\tau \in \mathcal{T}_f$, where Γ_f is the canonical obstruction for f . Recall that the action on any invariant stratum is given by pullbacks of complex structures by a collection of maps σ_{f^C} for all components C of any surface in the stratum. As we iterate, a new complex structure on each component is obtained by pulling back an old one from the image component. The combinatorics of the process is very simple: we have a map from a finite set into itself, every component is (pre-)periodic. The whole action, therefore, can be characterized by studying cycles of components. For each component C there are three cases (compare [Pil03, Canonical Decomposition Theorem]): the composition $F := F^C$ of all coverings in the cycle (that is the first-return map of C) is one of the following:

- a homeomorphism,

- a Thurston map with parabolic orbifold,
- a Thurston map with hyperbolic orbifold.

In the first case, the map σ_F acts on \mathcal{T}_F as an element of the mapping class group. We prove below that in the stratum \mathcal{S}_{Γ_f} all Thurston maps with hyperbolic orbifold are not obstructed.

By considering f^r where r is the least common multiple of the lengths of all cycles of components, we can assume that all components of C are fixed or prefixed. Clearly $P_f = P_{f^r}$ (which means that $\mathcal{T}_f = \mathcal{T}_{f^r}$), $\sigma_{f^r} = \sigma_f^r$, and any f -invariant multicurve Γ is f^r -invariant; the Thurston matrix M_Γ for f^r is the r -th power of the analogous Thurston matrix for f (see Lemma 1.1 in [DH93]). Moreover, the following is immediate.

Proposition 4.5.1. $\Gamma_f = \Gamma_{f^r}$.

Proof. Take any $\tau \in \mathcal{T}_f$. If $\gamma \in \Gamma_f$ then $l(\gamma, \sigma_f^n(\tau)) \rightarrow 0$. In particular, $l(\gamma, \sigma_f^{nr}(\tau)) = l(\gamma, \sigma_{f^r}^n(\tau)) \rightarrow 0$, hence $\gamma \in \Gamma_{f^r}$.

Set $D = d_T(\tau, \sigma_f(\tau))$. Since σ_f is weakly contracting and $\log l(\cdot, \gamma)$ is 1-Lipschitz, we have $l(\gamma, \sigma_f^{nr+s}(\tau)) \leq e^{Ds} l(\gamma, \sigma_f^{nr}(\tau))$. Therefore, if $l(\gamma, \sigma_f^{nr}(\tau)) \rightarrow 0$ then $l(\gamma, \sigma_f^n(\tau)) \rightarrow 0$ as well. \square

Using the tools developed, we are now able to give a proof of the following conjecture from [Pil03].

Theorem 4.5.2. *If the first-return map F of a periodic component C of the topological surface corresponding to the stratum \mathcal{S}_{Γ_f} has hyperbolic orbifold then F is not obstructed and, hence, equivalent to a rational map.*

Proof. We start again by considering the orbit of a point $\tau_0 \in \mathcal{T}_f$ and denote $\tau_n = \sigma_f^n(\tau)$, $m'_n = \pi_1(\tau_n)$, and $D = d_T(\tau_0, \tau_1)$. Theorems 3.2.3 and 3.2.4 imply that the limit set of m'_n is contained in a compact subset of $\mathcal{S}_{[\Gamma_f]} \subset \overline{\mathcal{M}'_f}$.

Any point $\hat{\tau}$ that lies in the stratum \mathcal{S}_{Γ_f} can be represented as $\hat{\tau} = (\hat{\tau}^1, \dots, \hat{\tau}^s)$ where $\hat{\tau}^i$ are points in the Teichmüller spaces corresponding to different components of the noded topological surface corresponding to the stratum; let us say that $\hat{\tau}^C := \hat{\tau}^1$ is the coordinate corresponding to C , i.e. a point in \mathcal{T}_F . We similarly write $\hat{m} = (\hat{m}^C, \dots, \hat{m}^s)$ for points in $\mathcal{S}_{[\Gamma_f]}$. As was explained above, we may assume that C is a fixed component. The action of σ on \mathcal{S}_{Γ_f} can then be written in a form $\sigma_f^i(\hat{\tau}) = (\sigma_F^i(\hat{\tau}^C), \dots)$. For notational convenience, we assume that $\|d\sigma_F\| < 1$;

otherwise we can take an appropriate iterate of f and work from there. Denote by π^C, π_1^C, π_2^C and $\tilde{\sigma}_F$ the maps that we get applying Proposition 4.3.1 to F where \mathcal{M}'_F is the “restriction” of \mathcal{M}'_f to the component corresponding to C (we chose \mathcal{M}'_F so that π_1^C is a restriction of π_1 ; we may be able to construct a smaller covering space for F that satisfies conditions of Propostion 4.3.1).

To illustrate the idea of the proof, let us first suppose that there exists an accumulation point $\hat{c} \in \mathcal{S}_{\Gamma_f}$ of $\{\tau_n\}$. Then, clearly, $\hat{c} := \sigma_f^{\hat{c}}(\hat{c})$ is also in the accumulation set of $\{\tau_n\}$ for all i . Hence, $\{\pi_1(\hat{c})\}$ is contained in a compact subset of $\mathcal{S}_{[\Gamma_f]}$. Considering the coordinate corresponding to C , we get that $\pi_1^C(\hat{c}) = \pi_1^C(\sigma_f^{\hat{c}}(\hat{c}))$ lie in a compact subset of \mathcal{M}'_F . Theorem 3.2.3 would imply that F is not obstructed. In the rest of the proof we will remove the assumption we made. For each $t \in \mathbb{N}$ we find a point $\hat{\tau}_k \in \mathcal{S}_{\Gamma_f}$ such that the first t elements of the sequence $\{\pi_1(\sigma_f^i(\hat{\tau}_k))\}$ coincide with the corresponding elements of some fixed sequence contained in a compact subset of \mathcal{M}'_f . From this we will be able to conclude the proof.

Choose a sequence $\{n_k^1\}$ so that $m'_{n_k^1} \rightarrow \hat{m}_0 \in \mathcal{S}_{[\Gamma_f]}$, then choose a subsequence $\{n_k^2\}$ of $\{n_k^1\}$ so that $m'_{n_k^2+1} \rightarrow \hat{m}_1$ and so on, so that $m'_{n_k^i+j} \rightarrow \hat{m}_j$ for all $0 \leq j \leq i$. Then the diagonal subsequence $\{n_k := n_k^k\}$ satisfies $m'_{n_k+i} \rightarrow \hat{m}_i$ for all $i = \overline{0, \infty}$. The commutative diagram (4.4) implies $\tilde{\sigma}_f(m'_{n_k+i}) = \pi_2(m'_{n_k+i+1})$ so by continuity (Theorem 1.0.2) we get

$$\tilde{\sigma}_f(\hat{m}_i) = \pi_2(\hat{m}_{i+1}). \quad (4.5)$$

Suppose $d_{WP}(m'_{n_k}, \hat{m}_0) < \varepsilon$ and $d_{WP}(m'_{n_k+1}, \hat{m}_1) < \varepsilon$ for some k , where $\varepsilon > 0$ is small. Then there exists a point $\hat{\tau}_k \in \mathcal{S}_{\Gamma_f}$ such that $d_{WP}(\tau_{n_k}, \hat{\tau}_k) = d_{WP}(m'_{n_k}, \hat{m}_0) < \varepsilon$ and $\pi_1(\hat{\tau}_k) = \hat{m}_0$. Then Proposition 3.3.2 implies

$$\begin{aligned} d_{WP}(\hat{m}_1, \pi_1(\sigma_f(\hat{\tau}_k))) &\leq d_{WP}(m'_{n_k+1}, \pi_1(\sigma_f(\hat{\tau}_k))) + d_{WP}(\hat{m}_1, m'_{n_k+1}) \leq \\ &\leq d_{WP}(\tau_{n_k+1}, \sigma_f(\hat{\tau}_k)) + \varepsilon \leq (\sqrt{d_f} + 1)\varepsilon. \end{aligned}$$

Since $\pi(\sigma_f(\hat{\tau}_k)) = \tilde{\sigma}_f(\hat{m}_0) = \pi_2(\hat{m}_1)$ we have that $\pi_1(\sigma_f(\hat{\tau}_k))$ lies in the fiber $\pi_2^{-1}(\pi_2(\hat{m}_1))$. Since $\pi_2^{-1}(\pi_2(\hat{m}_1))$ is finite in \mathcal{M}'_f there exists a positive constant c_1 such that the Weil-Petersson distance between any two different points in the fiber is at least c_1 . Similarly, set c_i to be the minimal distance between any two different points in the fiber $\pi_2^{-1}(\pi_2(\hat{m}_i))$ for all $i = \overline{1, \infty}$. We conclude that $\pi_1(\sigma_f(\hat{\tau}_k)) = \hat{m}_1$ if $\varepsilon \leq c_1/(\sqrt{d_f} + 1)$.

For any positive integer t , we set $\varepsilon(t) = \min_{i=\overline{1,t}}\{c_i\}/(\sqrt{d_f} + 1)$ and chose $k = k(t)$ large enough so that $d_{WP}(m'_{n_k+i}, \hat{m}_i) < \varepsilon(t)$ for all $i = \overline{1,t}$. Then the same

reasoning yields

$$\pi_1(\sigma_f^i(\hat{\tau}_k)) = \hat{m}_i \quad (4.6)$$

for all $i = \overline{1, t}$.

By the first part of Proposition 3.2.1 $d_{WP}(\tau_{n_k}, \tau_{n_{k+1}}) \leq C^0 d_T(\tau_{n_k}, \tau_{n_{k+1}})$ and therefore

$$\begin{aligned} d_{WP}(\hat{\tau}_k, \sigma_f(\hat{\tau}_k)) &\leq d_{WP}(\hat{\tau}_k, \tau_{n_k}) + d_{WP}(\tau_{n_k}, \tau_{n_{k+1}}) + d_{WP}(\tau_{n_{k+1}}, \sigma_f(\hat{\tau}_k)) \leq \\ &\leq \varepsilon(t) + C^0 \cdot D + \sqrt{d_f} \varepsilon(t) \leq c_1 + C^0 \cdot D. \end{aligned}$$

is bounded independently of k . The length L of the shortest simple closed geodesic on the Riemann surface corresponding to $\hat{\tau}_k^C$ is the same for all k since $\pi(\hat{\tau}_k) = \pi_2(\hat{m}_0)$ does not depend on k . Using the second part of Proposition 3.2.1 we obtain

$$\begin{aligned} d_T(\hat{\tau}_k^C, \sigma_F(\hat{\tau}_k^C)) &\leq C^1(L) d_{WP}(\hat{\tau}_k^C, \sigma_F(\hat{\tau}_k^C)) \leq \\ &\leq C^1(L) d_{WP}(\hat{\tau}_k, \sigma_f(\hat{\tau}_k)) \leq C^1(L)(c_1 + C^0 \cdot D) =: D_1 \end{aligned}$$

where D_1 is a constant (note that the first two distances are measured in \mathcal{T}_F ; the second inequality follows from the definition of the Weil-Petersson metric on the boundary, see Section 3.3). Then (4.6) implies $\pi_1^C(\sigma_F^i(\hat{\tau}_k^C)) = \hat{m}_i^C$ for $i = \overline{1, t}$, thus this sequence lies in a compact subset of \mathcal{M}'_F . From the fact that σ_F is uniformly contracting on the π_1^C -preimage of any compact set in \mathcal{M}'_F , it follows that $d_T(\hat{m}_i^C, \hat{m}_{i+1}^C) \leq d_T(\sigma_F^i(\hat{\tau}_k^C), \sigma_F^{i+1}(\hat{\tau}_k^C)) \leq q^i D_1$ for some $q < 1$ and all $i = \overline{1, t}$.

Since the bound D_1 does not depend on k , the inequality $d_T(\hat{m}_i^C, \hat{m}_{i+1}^C) \leq q^i D_1$ follows for all i and, hence, $\{\hat{m}_i^C\}$ converges to some $\hat{m}^C \in \mathcal{M}'_F$. It follows from (4.5) that $\tilde{\sigma}_F(\hat{m}^C) = \pi_2^C(\hat{m}^C)$ since all the maps involved are continuous. This means that for any point τ in the fiber $(\pi_1^C)^{-1}(\hat{m}^C)$ both τ and $\sigma_F(\tau)$ lie in $A = (\pi^C)^{-1}(\pi_2^C(\hat{m}^C))$. Let $R > 0$ be the lower bound on the Teichmüller distance in \mathcal{T}_F between distinct points in the fiber A .

Choose t and ε such that $2\varepsilon + q^t D_1 < R$ and $d_T(\hat{m}^C, \hat{m}_t^C) < \varepsilon$; set $k = k(t)$. Since $\pi_1^C(\sigma_F^t(\hat{\tau}_k^C)) = \hat{m}_t^C$ we can find a point $\hat{\tau}^C \in \mathcal{T}_F$ such that $\pi_1^C(\hat{\tau}^C) = \hat{m}^C$ and $d_T(\hat{\tau}^C, \sigma_F^t(\hat{\tau}_k^C)) = d_T(\hat{m}^C, \hat{m}_t^C) < \varepsilon$. Then

$$\begin{aligned} d_T(\hat{\tau}^C, \sigma_F(\hat{\tau}^C)) &\leq \\ &\leq d_T(\hat{\tau}^C, \sigma_F^t(\hat{\tau}_k^C)) + d_T(\sigma_F^t(\hat{\tau}_k^C), \sigma_F^{t+1}(\hat{\tau}_k^C)) + d_T(\sigma_F^{t+1}(\hat{\tau}_k^C), \sigma_F(\hat{\tau}^C)) \leq \\ &\leq \varepsilon + q^t D_1 + \varepsilon < R. \end{aligned}$$

Since both $\hat{\tau}^C$ and $\sigma_F(\hat{\tau}^C)$ belong to A , it follows that $\sigma_F(\hat{\tau}^C) = \hat{\tau}^C$ yielding that F

is equivalent to a rational map. \square

As a corollary we have the following

Theorem 4.5.3. *If the first-return maps of all periodic components of the topological surface corresponding to the stratum \mathcal{S}_{Γ_f} have hyperbolic orbifolds, then σ_f has a unique fixed point $\hat{\tau}$ in this stratum, and the orbit of any point in \mathcal{T}_f converges to $\hat{\tau}$.*

Proof. Take any point $\hat{\tau}_0$ in \mathcal{S}_{Γ_f} and consider its forward orbit. Since all F^C are not obstructed, the sequence will tend to a limit $\hat{\tau}$ in \mathcal{S}_{Γ_f} which is a fixed point by Theorem 1.0.2. Indeed, if C is a fixed component then we deal with Thurston's pullback map for a branched cover F^C which has hyperbolic orbifold and is not obstructed. Therefore the coordinate corresponding to this component converges to the unique fixed point of F^C . If C is in a cycle of components of length n , then by the same argument the coordinate corresponding to C for the sequence $\{\sigma_f^{nk+i}(\hat{\tau}_0)\}$ converges to the unique fixed point of F^C for any given i as k goes to infinity. Thus the coordinate corresponding to C converges for the whole sequence $\sigma_f^k(\hat{\tau}_0)$. Convergence of coordinates corresponding to pre-periodic components follows then from continuity of Thurston's pullback map.

To see that every orbit in \mathcal{T}_f converges to $\hat{\tau}$, note that from the proof of the previous theorem, it follows that $\hat{\tau}$ is in the limit set of any orbit in \mathcal{T}_f . On the other hand, it is easy to see that $\hat{\tau}$ is weakly attracting in the sense of the definition given in Section 4.2, therefore the orbit must converge to $\hat{\tau}$. \square

4.6 Topological characterization of canonical obstructions

Recall that the canonical obstruction of a Thurston map f is defined as the set of all homotopy classes of curves for which the length of the corresponding geodesics tends to 0 as we iterate in the Teichmüller space T_f . Notice that this definition makes sense in the case of Thurston maps with parabolic orbifolds. However, only one of the implications in Theorem 3.2.3 is true in this setting.

Proposition 4.6.1. *If a Thurston map f with a parabolic orbifold is Thurston equivalent to a rational map then its canonical obstruction Γ_f is empty.*

Proof. If we start iterating σ_f at a fixed point τ then, obviously, the lengths of all geodesics are uniformly bounded from below. \square

The other direction of Theorem 3.2.3 tells that if for a Thurston map with hyperbolic orbifold the canonical obstruction is empty then there exist no obstructions at all for this map. In the general case, the following is true.

Theorem 4.6.2. *Suppose that the canonical obstruction of a Thurston map f is empty, and Γ is a simple Thurston obstruction for f . Then f is a $(2, 2, 2, 2)$ -map and every curve of Γ has two postcritical points of f in each complementary component.*

Proof. The map f must have parabolic orbifold by Theorem 3.2.3. Since the canonical obstruction is empty, the leading eigenvalue of M_Γ is equal to 1.

Case I. As an illustration of the idea of the proof, we first consider the case when Γ consists of a single simple closed curve γ . Then the Thurston matrix has the form $M_\Gamma = (1)$. Denote by $r(\tau) = M(\gamma, \tau)$ the maximal modulus of an annulus homotopic to γ in the Riemann surface corresponding to τ . (We can always find an annulus of maximal modulus by Lemma 2.4.6.) Recall, that by the Collar Lemma, this is approximately equal to $1/l(\gamma, \tau)$ (see Theorem 2.4.3). Then, by the Grötzsch inequality (see more detailed explanation below), $r(\sigma_f(\tau)) \geq r(\tau)$. As usual, denote $\tau_n = \sigma_f^n(\tau_0)$. The sequence $r_n = r(\tau_n)$ is increasing and bounded because γ is not a part of canonical obstruction and, therefore, $l(\gamma, \tau_n) > L > 0$ for some L . It follows that r_n has a limit which we denote by r .

We pick initial τ_0 so that r_0 is large enough to make sure that the geodesic corresponding to γ is shorter than any simple closed geodesic that intersects γ . By Corollary 2.4.5, it is enough to take τ_0 such that $r_0 > \pi/\ln(3 + 2\sqrt{2})$.

Theorem 3.2.6 implies that all $m'_n = \pi_1(\tau_n)$ belong to a compact subset of \mathcal{M}'_f , where π_1 is defined as before (see Proposition 4.3.1). Consider a subsequence n_k such that $\{m'_{n_k}\}$ converges to $p \in \mathcal{M}'_f$ and $\{m'_{n_k+1}\}$ converges to $q \in \mathcal{M}'_f$. On either of Riemann surfaces corresponding to p and q , there exists exactly one homotopy class of simple closed curves γ' such that γ' and γ are in the same equivalence class under the action of the covering group corresponding to π_1 , and that $M(\gamma', p) = r$ (or $M(\gamma', q) = r$ respectively). Indeed, any two different curves in the same equivalence class have non-zero intersection number because otherwise they would bound an annulus that cannot contain any of the marked points and, thus, be homotopic to each other. Since $r > r_0$ is large enough, $M(\gamma'', p) < r$ (and the same for q) for any other curve γ'' in the equivalence class of γ because otherwise γ' and γ'' would not intersect by the Collar Lemma. The same statement obviously holds for equivalence classes of curves with respect to the pure mapping class group.

Recall the commutative diagram (4.4):

$$\begin{array}{ccc} \mathcal{T}_f & \xrightarrow{\sigma_f} & \mathcal{T}_f \\ \pi_1 \downarrow & & \downarrow \pi \\ \mathcal{M}'_f & \xrightarrow{\tilde{\sigma}_f} & \mathcal{M}_f \end{array}$$

Since all the maps involved are continuous, it follows that $\tilde{\sigma}_f(p) = \lim \tilde{\sigma}_f(m'_{n_k}) = \lim \pi(\tau_{n_k+1}) = \pi_2(q)$. Take any point $\tau \in \mathcal{T}_f$ such that $\pi_1(\tau) = p$ and $r(\tau) = M(\gamma, \tau) = r$ (a point p provides a conformal structure on the base topological surface together with marking of some equivalence classes of curves, to get a point in the fiber $\pi_1^{-1}(p)$ we simply choose a representative of each marked class; for the equivalence class of γ we choose the shortest representative). Then from the same diagram we get $\pi(\sigma_f(\tau)) = \pi_2(q)$. Therefore $r(\sigma_f(\tau))$ is bounded above by $M(\gamma', \pi_2(q))$ where γ' is equivalent to γ under the action of the pure mapping class group. As we have seen above, this implies that $r(\sigma_f(\tau)) \leq r$. On the other hand, we know that $r(\sigma_f(\tau)) \geq r(\tau) = r$ which implies $r(\sigma_f(\tau)) = r(\tau) = r$. We now investigate under which conditions the inequality $r(\sigma_f(\tau)) \geq r(\tau)$ can become an equality.

Denote by A an annulus in the Riemann surface corresponding to τ that is homotopic to γ and has maximal possible modulus r (see Lemma 2.4.6); let annuli A_1, \dots, A_k be the disjoint preimages of A in the Riemann surface corresponding to $\sigma_f(\tau)$ under the map f_τ that are homotopic to γ . Each A_i is mapped to A by f_τ as a non-ramified cover of degree d_i . Therefore, $\text{mod } A_i = 1/d_i \text{ mod } A = r/d_i$. By the definition of Thurston obstructions, $M_{\{\gamma\}} = (\sum 1/d_i)$ hence $\sum 1/d_i = 1$. Consider the annulus B containing all A_i and homotopic to γ . By the Grötzsch inequality, $\text{mod } B \geq \sum \text{mod } A_i = \sum (1/d_i)r = r$.

If we assume that $r(\sigma_f(\tau)) = r(\tau)$ then $\text{mod } B \leq r$, hence B is an annulus homotopic to γ with maximal possible modulus and the last inequality is, in fact, an equality. Hence, the closure of B must be the whole Riemann sphere; and A_i are round subannuli of B and their closure covers B . If there were only one preimage A_1 of A , then the degree d_1 would have to be equal to 1; this would imply that f is not a Thurston map but a homeomorphism because the closure of $B = A_1$ is the whole sphere. Every two adjacent annuli share a common boundary component α which is an analytic curve. Then the corresponding boundary component $\beta = f_\tau(\alpha)$ of B is piece-wise (except for possibly at the images of critical points) smooth. Since both adjacent to α annuli are mapped to A , β has only one complementary component. We conclude that β is a smooth curve segment connecting two critical values of f

and passing through no other critical value, because a component of the preimage of β is a simple closed curve. Moreover, all critical points of f_τ in α are of degree 2. If the number of annuli k is at least 3 then the same is true for the other boundary component δ of A . Hence there exist at least 4 critical values – at least two in each complementary component of A – and this yields the statement of the theorem. We can reduce the case $k = 2$ to the case, say, $k = 4$ by considering the second iterate of f . Alternatively, we can prove it directly as follows.

The number of critical points on α is equal to twice the degree of f on either of the annuli; in particular, this yields d_i are all equal. Thus, in the case of exactly two annuli, the map f has degree 4. We already know that there are exactly 4 critical points of f_τ on α . Each of the two preimages δ_1 and δ_2 of δ must contain at least one critical point because otherwise $A_1 \cup \delta_1$ (or $A_2 \cup \delta_2$ respectively) contains no critical points and, hence, maps by f_τ conformally on $A \cup \delta$ which is a contradiction. Therefore, f has exactly 6 simple critical points since a branched cover of degree 4 has at most $4 * 2 - 2 = 6$ critical points. This implies that f is a $(2, 2, 2, 2)$ -map. Indeed, the degree of f is 4 so every critical value is the image of at most two different critical points. Thus, we must have at least 3 critical values and their full preimage consists of the 6 critical points. If the postcritical set contains only 3 points, then all critical values are also critical points and the signature of the orbifold corresponding to f is (∞, ∞, ∞) , which contradicts the fact that f has parabolic orbifold. We see that \mathcal{P}_f contains at least 4 points; since f has parabolic orbifold we conclude that f is a $(2, 2, 2, 2)$ -map.

Note that the endpoints of β are the only two postcritical points of f on β , because every other point on β has exactly 4 preimages on α and none of them are either critical or marked. Therefore, each complementary component of γ has two postcritical points.

Case II. We use the same approach in the case when $M_\Gamma = \{\gamma_1, \dots, \gamma_k\}$ is positive and $k \geq 2$. By the Perron-Frobenius theorem (Theorem 2.6.1), there exists a positive eigenvector v corresponding to $\lambda_\Gamma = 1$. Denote $r(\tau) = \sup \min_{i=\overline{1, k}} \text{mod } A_i/v_i$, where the supremum is taken over all configurations of disjoint annuli A_i in the Riemann surface corresponding to τ , such that A_i is homotopic to γ_i for $i = \overline{1, k}$.

Consider annuli B_i in the Riemann surface corresponding to $\sigma_f(\tau)$, such that B_i is homotopic to γ_i and B_i contains all preimages of A_j that are homotopic to γ_i for every $i = \overline{1, k}$. The same arguments as above show that $(\text{mod } B_i)^T \geq M_\Gamma(\text{mod } A_i)^T$. If A_i is the maximal configuration that realizes $r(\tau)$ (again, existence of such a configuration follows from Lemma 2.4.6), then, by definition, $(\text{mod } A_i)^T \geq r(\tau)v$

and, hence, $(\text{mod } B_i)^T \geq M_\Gamma r(\tau)v = r(\tau)v$. Thus, $r(\sigma_f(\tau)) \geq r(\tau)$. We use precompactness of m'_n as above to construct a point τ such that $r(\sigma_f^2(\tau)) = r(\sigma_f(\tau)) = r(\tau) = r$. This means that there exists i for which $\text{mod } B_i = rv_i$ and the inequality corresponding to the i -th line of $(\text{mod } B_i)^T \geq M_\Gamma(\text{mod } A_i)^T$ is an equality. Since all entries of M_Γ are positive, this immediately forces $\text{mod } A_i = rv_i$ for all i . Because $r(\sigma_f^2(\tau)) = r$, the same reasoning implies that $\text{mod } B_i = rv_i$, for all i , and $\{B_i\}$ is a maximal configuration for $\sigma_f(\tau)$. In particular, we infer that the closure of the union of B_i covers the whole sphere, and Γ is invariant and no curve of Γ has a non-essential preimage.

Moreover, all f_τ -preimages of annuli A_j that are homotopic to γ_i are round subannuli of B_i and closure of their union covers B_i . If two preimages of some A_j abut along a boundary component then the image boundary component of A_j is a smooth curve segment (see above). If preimages of two different A_j and A_l abut in $\sigma_f(\tau)$ then the annuli themselves abut in τ . Since for every j , there exists at least one preimage of A_j that is homotopic to γ_i , all annuli are concentric and so are the curves γ_i . Since B_i are homotopic to γ_i , it follows that all preimages of A_i are also concentric. We see that no boundary component that separates two annuli A_i can contain a postcritical point. By removing marked points on these boundary components, we can reduce the statement of the theorem to the previous case.

Case III. The general case is now easily reduced to the previous case. Suppose Γ is an arbitrary simple obstruction. Take a subset Γ_1 such that M_{Γ_1} is irreducible and $\lambda_{\Gamma_1} = 1$ (see Section 2.6). By Theorem 2.6.5, some iterate k of M_{Γ_1} can be conjugated by a permutation matrix into the block form where all the blocks on diagonal are positive and all other blocks are zero, moreover, the leading eigenvalue of each block is 1. Take further subset $\Gamma_2 \subset \Gamma_1$ corresponding to any of the diagonal blocks. Then M_{Γ_2} with respect to f^k is a positive matrix with leading eigenvalue 1. From the previous cases, we know that f^k is a $(2, 2, 2, 2)$ -map and that every element of Γ_2 has two postcritical points in each complementary component. This immediately implies, that f itself is a $(2, 2, 2, 2)$ -map with the same postcritical set. Moreover, we get that Γ_2 is invariant. Therefore, Γ_1 is a union of invariant multicurves for f^k and no curve of these multicurves has a non-essential preimage. It follows that Γ_1 is f -invariant and all curves in Γ_1 have two postcritical points in each complementary component.

Set $\Gamma' = \Gamma \setminus \Gamma_1$. Then $\lambda_{\Gamma'} = 1$ because Γ is simple and Γ_1 is invariant. We repeat the argument until we exhaust Γ , proving the desired property for all curves in Γ . \square

Recall that to a $(2, 2, 2, 2)$ -map f we associate a linear operator $\hat{f}_*: H_1(T, \mathbb{Z}) \rightarrow H_1(T, \mathbb{Z})$ (see Section 3.5). The following statement is in a certain sense a converse to the previous theorem.

Theorem 4.6.3. *The canonical obstruction of a $(2, 2, 2, 2)$ -map f contains a curve that has two postcritical points in each complementary component if and only if \hat{f}_* has two different integer eigenvalues.*

Proof. Let \hat{f}_* have two different integer eigenvalues d_1 and d_2 . Without loss of generality, we assume that both are positive and $d_1 < d_2$. We first assume that P_f has exactly four points. Take a curve γ in $\mathbb{S}^2 \setminus P_f$ such that $\langle \gamma \rangle$ is an eigenvector of \hat{f}_* corresponding to d_1 . For any connected component γ' of the f -preimage of γ , we have $\hat{f}_*(\langle \gamma' \rangle) = d \langle \gamma \rangle$ with some $d \in \mathbb{Z}$. It follows that $\langle \gamma' \rangle$ is also an eigenvector of \hat{f}_* corresponding to d_1 because \hat{f}_* is diagonalizable. We infer that all preimages of γ are homotopic to γ and each of them is mapped to γ by f with degree d_1 . Since the degree of f is equal to $d_1 d_2$, there are exactly d_2 preimages of γ . Therefore, $\Gamma = \{\gamma\}$ is an obstruction with $M_\Gamma = (d_2/d_1)$ and $\Gamma \subset \Gamma_f$ because the leading eigenvalue $\lambda_\Gamma = d_2/d_1 > 1$.

If P_f has more than four points, we first consider the Thurston map $F = (f, \mathcal{P}_f)$ where \mathcal{P}_f is the postcritical set of f . By the arguments above, there exists a curve γ that has two postcritical points in each complementary component which is a part of the canonical obstruction of F . Clearly, there exists a canonical projection p from \mathcal{T}_f to \mathcal{T}_F that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{T}_f & \xrightarrow{\sigma_f} & \mathcal{T}_f \\ p \downarrow & & p \downarrow \\ \mathcal{T}_F & \xrightarrow{\sigma_F} & \mathcal{T}_F \end{array}$$

As we iterate $p(\tau_1)$ in the Teichmüller space \mathcal{T}_F , the maximal modulus $M(\gamma, p(\tau_m))$ of an annulus homotopic to γ tends to infinity. This is the same as the maximal modulus on an annulus A homotopic to γ in $\tau_m \in \mathcal{T}_f$ if we fill in all extra punctures. The extra marked points split A into at most $k + 1$ concentric annuli, where k is the number of points in $P_f \setminus \mathcal{P}_f$. Thus, the lengths of all homotopy classes of curves in $\mathbb{S}^2 \setminus P_f$ that are homotopic to γ relative \mathcal{P}_f cannot be uniformly bounded from below as we iterate. Indeed, in this case all concentric annuli must have modulus uniformly bounded from above, and since there are at most $k + 1$ of them, the modulus of an annulus homotopic to γ relative \mathcal{P}_f would be also bounded from above, which is a contradiction. By Theorem 3.2.6, there exists a curve homotopic to γ relative P'_f in

the canonical obstruction Γ_f .

Let us now prove the converse direction; suppose that γ is a curve in the canonical obstruction of f that has two postcritical points in each complementary component. By Corollary 4.3.3, the canonical obstruction Γ_f is invariant. This implies that every component of the f -preimage of γ has zero intersection number with γ . Therefore, $\langle \gamma \rangle$ is an eigenvector of \hat{f}_* corresponding to some eigenvalue $d \in \mathbb{Z}$ by Proposition 3.5.4. If we identify $H_1(T, \mathbb{Z})$ with \mathbb{Z}^2 , then \hat{f}_* becomes a multiplication by an integer 2×2 matrix. We see that the other eigenvalue a of \hat{f}_* is also an integer. We suppose, by contradiction, that the other eigenvalue of \hat{f}_* is also equal to d . We fix a basis of $H_1(T, \mathbb{Z})$ containing $\langle \gamma \rangle$, the operator \hat{f}_* then assumes the form

$$\hat{f}_* = \begin{pmatrix} d & a \\ 0 & d \end{pmatrix},$$

where $a \in \mathbb{Z}$. As above, we consider the Thurston map $F = (f, \mathcal{P}_f)$ instead of f . Then γ (viewed as a curve on $\mathbb{S}^2 \setminus \mathcal{P}_f$) must lie in the canonical obstruction of F .

If $a = 0$, then \hat{f}_* is just a scalar multiplication by d so any element of $H_1(T, \mathbb{Z})$ is an eigenvector. Take any γ' such that γ' is not homotopic to γ relative \mathcal{P}_f , and there are two postcritical points of f in each complementary component of γ' . Since $\langle \gamma' \rangle$ is an eigenvector of \hat{f}_* , we see that there exist d components of the preimage of γ' that are homotopic to γ' and are mapped to γ' with degree d . Thus, the multicurve consisting of a single curve γ' is an obstruction for F and, hence, γ' cannot have positive intersection with any curve in the canonical obstruction of F , in particular, with γ . This is clearly a contradiction because \mathcal{P}_f has only four points.

Similar argument also works in the case $a \neq 0$. We may assume that $a = d \cdot b$ with $b \in \mathbb{Z}$, otherwise we can consider F^d , for which γ will be still in the canonical obstruction, but the action on $H_1(T, \mathbb{Z})$ will be given by

$$\hat{f}_*^d = \begin{pmatrix} d & a \\ 0 & d \end{pmatrix}^d = \begin{pmatrix} d^d & d^d a \\ 0 & d^d \end{pmatrix}.$$

Consider a curve γ_0 such that $\langle \gamma_0 \rangle = (0, 1)^T$. Then for a component γ_1 of the preimage of γ_0 we have $\hat{f}_*(\langle \gamma_1 \rangle) = d_0 \langle \gamma_0 \rangle = (0, d_0)^T$ for some $d_0 \in \mathbb{Z}$. The coordinates x, y of $\langle \gamma_1 \rangle = (x, y)^T$ must be co-prime since γ_1 is a simple closed curve. Therefore, from

$$\hat{f}_* \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d & d \cdot b \\ 0 & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d(x + by) \\ dy \end{pmatrix} = \begin{pmatrix} 0 \\ d_0 \end{pmatrix}$$

we infer that $y = 1$ and $d_0 = d$. (Recall that coordinates of $\langle \gamma_1 \rangle$ are defined up to sign, depending on the chosen orientation; we can, hence, force y to be positive.) As above, it follows that γ_0 has exactly d preimages that are all homotopic to γ_1 and mapped to γ_1 with degree d , where $\langle \gamma_1 \rangle = (-b, 1)^T$. By induction, the same statement is true for all $i \in \mathbb{N}$ with $\langle \gamma_i \rangle = (-i \cdot b, 1)^T$.

The curves γ_i form an infinite analogue of a Thurston obstruction; the corresponding linear transformation M_Γ acts on $\mathbb{R}^{\mathbb{N}}$ by shifting the indexes of basis vectors. By the same reasoning as above, we get $M(\gamma_{i+1}, \tau_{i+1}) \geq M(\gamma_i, \tau_i)$ for all $i \in \mathbb{N}$. Since all γ_i have positive intersection with γ , we conclude that γ cannot be in the canonical obstruction of F , which is a contradiction. \square

Corollary 4.6.4. *The canonical obstruction of a $(2, 2, 2, 2)$ -map f is empty if and only if every curve of every simple Thurston obstruction for f has two postcritical points of f in each complementary component and the two eigenvalues of \hat{f}_* are equal or non-integer. For any other Thurston map, the canonical obstruction is empty if and only if there exist no Thurston obstruction for f .*

Proof. The statement follows directly from the previous two theorems. \square

The following theorem is a generalization of Pilgrim's conjecture (Theorem 4.5.2). We use a different approach here.

Theorem 4.6.5. *If the first-return map F of a periodic component C of the noded topological surface corresponding to pinching curves in Γ_f is a Thurston map then the canonical obstruction of F is empty.*

Proof. The main idea of the proof is essentially the same as in the proof of Theorem 4.6.2. Suppose, on contrary, that $\Gamma \neq \emptyset$ is the canonical obstruction of F . Passing to an appropriate iterate of f , we may assume that C is a fixed component (see Proposition 4.5.1). In this case, Γ is f -invariant. Since Γ is not a part of the canonical obstruction, the leading eigenvalue λ_Γ is equal to 1. Taking a higher iterate of f , if needed, we can find a subset $\Gamma' \subset \Gamma$ such that $\lambda_{\Gamma'} = 1$ and $M_{\Gamma'}$ is positive.

We can repeat the proof of Case II of Theorem 4.6.2 almost verbatim. As before, there exists a positive eigenvector v corresponding to the leading eigenvalue 1. Denote $r(\tau) = \sup \min_{i=1, \dots, k} \text{mod } A_i / v_i$, where the supremum is taken over all configurations of disjoint annuli A_i in the Riemann surface corresponding to τ , such that A_i is homotopic to $\gamma_i \in \Gamma'$. We have already established that $r(\sigma_f(\tau)) \geq r(\tau)$. Take an accumulation point m' of the projection of an arbitrary orbit to the space

\mathcal{M}' . Then $m' \in \mathcal{S}_{\Gamma_f}$ by Theorem 3.2.6 and $r(m') = r(\tilde{\sigma}_f(m'))$. Since all annuli that are homotopic to curves in Γ' on the noded surface corresponding to any noded surface in \mathcal{S}_{Γ_f} must be contained in C , from this point on the proof goes the same way.

We conclude that F is a $(2, 2, 2, 2)$ -map and all curves of Γ' have two postcritical points of F in each complementary component. However, by the previous theorem, no such curve can be contained in the canonical obstruction of F . Otherwise, the leading eigenvalue of Γ' would be strictly greater than 1, which would force Γ' to be a part of the canonical obstruction of f . This contradiction shows that the canonical obstruction of F is empty. \square

We are now in the position to formulate and prove a pure topological criterion that singles out canonical obstructions. It says that the canonical obstruction is the minimal obstruction satisfying the conclusion of the previous theorem.

Theorem 4.6.6 (Characterization of Canonical Thurston Obstructions). *The canonical obstruction Γ is a unique minimal Thurston obstruction with the following properties.*

- *If the first-return map F of a cycle of components in \mathcal{S}_{Γ} is a $(2, 2, 2, 2)$ -map, then every curve of every simple Thurston obstruction for F has two postcritical points of f in each complementary component and the two eigenvalues of \hat{F}_* are equal or non-integer.*
- *If the first-return map F of a cycle of components in \mathcal{S}_{Γ} is not a $(2, 2, 2, 2)$ -map or a homeomorphism, then there exists no Thurston obstruction of F .*

Proof. By Corollary 4.6.4, both conditions above are equivalent to saying that the canonical obstruction for F is empty. The necessity of these conditions then follows from Theorem 4.6.5. Suppose, on contrary, that Γ is not minimal with these properties, i.e. there exists $\Gamma' \subsetneq \Gamma$ satisfying the same condition. Since Γ is simple by Corollary 4.3.3, at least one curve γ of $\Gamma \setminus \Gamma'$ must lie in a periodic component C of $\mathcal{S}_{\Gamma'}$. Consider the multicurve Γ_1 containing all curves of $\Gamma \setminus \Gamma'$ that lie in C ; as Γ is simple, Γ_1 is an obstruction for F . There are three cases.

Case I. The first-return map to C is a homeomorphism. Since γ is essential and not homotopic to any nodes, the component C must have at least four marked points. Take any other simple closed curve α_1 in C that has non-zero intersection with γ . Since the first-return map is a homeomorphism, γ is a part of a Levy cycle. Denote by α_2 the one-to-one preimage of α_1 and so on. Then, as in the proof of

Theorem 4.6.3, we see that $M(\alpha_{i+1}, \tau_{i+1}) \geq M(\alpha_i, \tau_i)$ and α_{ki} intersects γ for all i , where k is the length of the Levy cycle. This implies that the length of γ is bounded from above, which contradicts the assumption that γ was a part of the canonical obstruction.

Case II. The first-return map to C is a $(2, 2, 2, 2)$ -map. It follows that every curve in Γ_1 has two postcritical points of F in each complementary component. By the same argument as in the proof of Theorem 4.6.3, the length of these curves cannot tend to zero as we iterate f so Γ_1 cannot be a part of the canonical obstruction.

Case III. The first-return map to C is neither of the two cases above. By Corollary 4.6.4, F has no obstructions at all, hence Γ_1 must be empty.

Since any curve of any other obstruction either lies in Γ , or does not intersect any curve in Γ , the uniqueness of a minimal obstruction satisfying the conditions of the theorem follows from the same argument. \square

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