4/7/2020 R03

Tuesday, April 7, 2020 5:30 PM

Announcements: See Trig notes (posted online) for integrating $\int \sin^k x \cdot dx$ or $\int \cos^k x \cdot dx$ when k even

need to use double-angle formulas.

MLC: go to math.stonybrook.edu/mlc
click on "Center Hours"
I will be online Mon 2-3pm
Th 5:30-6:30pm

OH: Th 4:30-5:30pm

Integrating $\int \tan^k x \cdot \sec^j x \cdot dx$. (see guidelines on pg. 278 of Calculus Vol. 2)

Important cases:

(a) $j \geq 2$ and j even.

rewrite $\sec^j x = \underbrace{\sec^{j-2} x}_{} \cdot \boxed{\sec^2 x}$

rewrite in terms of $\tan x$ using $\sec^2 x = \tan^2 x + 1$.

[Pythorean identities:

$\sin^2 x + \cos^2 x = 1$

divide both sides by $\cos^2 x$: $\tan^2 x + 1 = \sec^2 x$.

divide both sides by $\sin^2 x$: $1 + \cot^2 x = \csc^2 x$]

Use u-sub with $u = \tan x$
$du = \sec^2 x \cdot dx$

Ex $\int \tan^3 x \cdot \sec^4 x \cdot dx$

$= \int \tan^3 x \cdot \underbrace{\sec^2 x}_{(\tan^2 x + 1)} \cdot \boxed{\sec^2 x \cdot dx}$ du

Apply u-sub
$u = \tan x$
$du = \sec^2 x \cdot dx$

$= \int u^3 \cdot (u^2 + 1) \cdot du$

$= \cdots$ integrate using power rule.

(b) $\int \tan^k x \cdot \sec^j x \cdot dx$ where k is odd, $j \geq 1$.

Strategy: rewrite $\tan^k x \cdot \sec^j x$
$= \tan^{k-1} x \cdot \sec^{j-1} x \cdot \underbrace{\tan x \cdot \sec x}_{\text{"du"}}$.

rewrite in terms of $\sec x$
$\tan^2 x = \sec^2 x - 1$.

Apply u-sub with $u = \sec x$
$du = \sec x \cdot \tan x \cdot dx$.

Ex $\int \tan^5 x \cdot \sec^2 x \cdot dx$

$= \int \tan^4 x \cdot \sec x \cdot \tan x \cdot \sec x \cdot dx$

$(\tan^2 x)^2 = (\sec^2 x - 1)^2$

$= \int (\sec^2 x - 1)^2 \cdot \sec x \cdot \boxed{\tan x \cdot \sec x \cdot dx}$ = du

u-sub:
$u = \sec x$

$= \int (u^2 - 1)^2 \cdot u \cdot du$

$= \cdots$ integrate using power rule

Exceptions: $\int \tan^k x \, dx$ where $k \geq 3$, k odd,

Ex $\int \tan^3 x \cdot dx$ use $\tan^2 x = \sec^2 x - 1$.

$= \int (\sec^2 x - 1) \cdot \tan x \cdot dx$

$= \int \sec^2 x \cdot \tan x \cdot dx - \boxed{\int \tan x \cdot dx}$ $= -\ln|\cos x|$

use u-sub:
$u = \tan x$, $du = \sec^2 x \, dx$

Ex $\int \sec^3 x \, dx$

$= \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x \cdot dx}_{dv}$

Integration by parts:
$u = \sec x$
$dv = \sec^2 x \cdot dx$
$du = \sec x \cdot \tan x \cdot dx$
$v = \tan x$

$= u \cdot v - \int v \cdot du$

$= \tan x \cdot \sec x - \int \underbrace{\tan x \cdot \sec x \cdot \tan x}_{\tan^2 x \cdot \sec x} \cdot dx$

distribute $(\sec^2 x - 1)$

$= \tan x \cdot \sec x - \int (\sec^2 x - 1) \cdot \sec x \cdot dx$

$= \tan x \cdot \sec x + \underbrace{\int \sec x \cdot dx}_{\ln|\sec x + \tan x|} - \int \sec^3 x \cdot dx = \int \sec^3 x \cdot dx$

add $\int \sec^3 x \, dx$ to both sides

$\Rightarrow 2 \cdot \int \sec^3 x \cdot dx = \tan x \cdot \sec x + \ln|\sec x + \tan x|$

$\Rightarrow \int \sec^3 x \cdot dx = \frac{1}{2} \tan x \cdot \sec x + \frac{1}{2} \ln|\sec x + \tan x|$.

Trig substitution: first try u-sub

Ex $\int \frac{x}{\sqrt{1-x^2}} dx$ ← can solve using u-sub

I. Integrals involving $\sqrt{a^2 - x^2}$

* use $x = a \cdot \sin\theta$, $dx = a \cdot \cos\theta \cdot d\theta$

Notice: $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \cdot \sin^2\theta} = a \cdot \sqrt{1 - \sin^2\theta}$
$= a \cdot \cos\theta$

Ex $\int x^3 \cdot \sqrt{1 - x^2} \cdot dx$ use $x = \sin\theta$, $dx = \cos\theta \cdot d\theta$

$= \int \sin^3\theta \cdot \cos\theta \cdot \cos\theta \cdot d\theta$

$= \int \sin^3\theta \cdot \cos^2\theta \cdot d\theta$

$= \int \underbrace{\sin^2\theta}_{1-\cos^2\theta} \cdot \cos^2\theta \cdot \sin\theta \cdot d\theta$

$= \int (1 - \cos^2\theta) \cdot \cos^2\theta \cdot \boxed{\sin\theta \, d\theta}$ = −du

Apply u-sub
$u = \cos\theta$
$du = -\sin\theta \cdot d\theta$

$= \cdots$ integrate polynomial.

II. Integrals involving $\sqrt{a^2 + x^2}$

* Use $x = a \cdot \tan\theta$, $dx = a \cdot \sec^2\theta \cdot d\theta$

Ex. $\int \frac{x^3}{\sqrt{1+x^2}} dx$ use $x = \tan\theta$, $dx = \sec^2\theta \cdot d\theta$

$= \int \frac{\tan^3\theta}{\sqrt{\underbrace{1 + \tan^2\theta}_{\sec^2\theta}}} \cdot \sec^2\theta \cdot d\theta$

$= \int \frac{\tan^3\theta \cdot \sec^{\cancel{2}}\theta}{\cancel{\sec\theta}} \cdot d\theta$

$= \int \tan^3\theta \cdot \sec\theta \cdot d\theta$ use u-sub (see first few examples)

$= \int \tan^2\theta \cdot \tan\theta \cdot \sec\theta \cdot d\theta$

$= \int (\sec^2\theta - 1) \cdot \tan\theta \cdot \sec\theta \cdot d\theta$

use u-sub
$u = \sec\theta$, $du = \tan\theta \cdot \sec\theta \cdot d\theta$

$= \int (u^2 - 1) \cdot du$

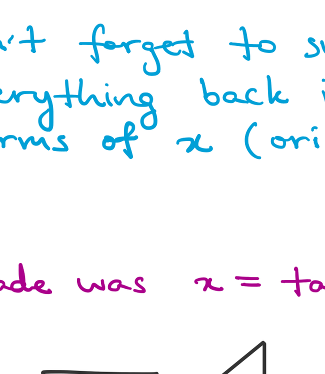
$= \frac{1}{3} \cdot u^3 - u + C$

don't forget to substitute everything back in. Write in terms of x (original variable)

$= \frac{1}{3} \cdot \sec^3\theta - \sec\theta + C$

[The first substitution we made was $x = \tan\theta$.

So $\tan\theta = \frac{x}{1} = \frac{\text{opp}}{\text{adj}}$



$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj.}} = \frac{\sqrt{1+x^2}}{1} = (1+x^2)^{1/2}$]

$= \frac{1}{3} \cdot (1+x^2)^{3/2} - (1+x^2)^{1/2} + C$.

III. Integrals involving $\sqrt{x^2 - a^2}$

* use $x = a \cdot \sec\theta$, $dx = a \cdot \sec\theta \cdot \tan\theta \cdot d\theta$

Ex. $\int \frac{dx}{\sqrt{x^2 - 4}}$ use $x = 2 \cdot \sec\theta$
$dx = 2 \cdot \sec\theta \cdot \tan\theta \cdot d\theta$

$= \int \frac{1}{\sqrt{4 \cdot \sec^2\theta - 4}} \cdot 2 \cdot \sec\theta \cdot \tan\theta \cdot d\theta$

$= \int \frac{1}{\sqrt{4(\underbrace{\sec^2\theta - 1}_{\tan^2\theta})}} \cdot 2 \cdot \sec\theta \cdot \tan\theta \cdot d\theta$

$= \int \frac{1}{\cancel{2} \cdot \cancel{\tan\theta}} \cdot \cancel{2} \cdot \sec\theta \cdot \cancel{\tan\theta} \cdot d\theta$

$= \int \sec\theta \cdot d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

[Recall $x = 2 \cdot \sec\theta$

So $\sec\theta = \frac{x}{2} = \frac{\text{hyp}}{\text{adj}}$

Hence $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2 - 4}}{2}$]

$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2}\right| + C$.

Ex (Challenge – first try on your own)

Find area between $f(x) = \sqrt{x^2 - 4}$ and x-axis over interval $[2, 5]$

Solution: we want to find
$\int_2^5 \sqrt{x^2 - 4} \, dx$ use $x = 2 \cdot \sec\theta$
$dx = 2 \cdot \tan\theta \cdot \sec\theta \cdot d\theta$

$= \int_a^b \underbrace{\sqrt{4 \cdot \sec^2\theta - 4}}_{2 \cdot \sqrt{\tan^2\theta} = 2 \cdot \tan\theta} \cdot 2 \cdot \tan\theta \cdot \sec\theta \cdot d\theta$

Bounds of integration change!
$2 = 2 \cdot \sec a \Rightarrow \sec a = 1 \Rightarrow \cos a = 1$
$\Rightarrow a = \sec^{-1}(1) = 0$.
$5 = 2 \cdot \sec b \Rightarrow \sec b = \frac{5}{2}$
$\Rightarrow b = \sec^{-1}(\frac{5}{2})$

$= \int_0^{\sec^{-1}(\frac{5}{2})} 4 \cdot \tan^2\theta \cdot \sec\theta \cdot d\theta$

$= 4 \cdot \int_0^{\sec^{-1}(\frac{5}{2})} (\sec^2\theta - 1) \cdot \sec\theta \cdot d\theta$

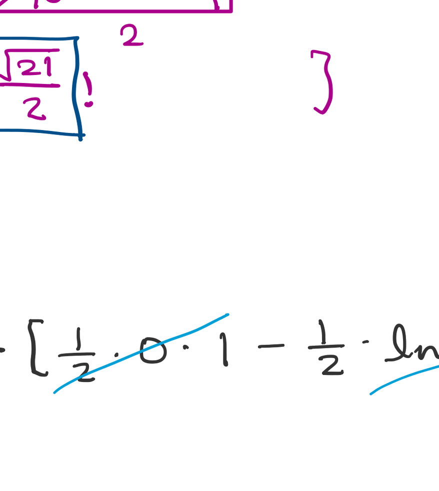
$= 4 \cdot \int_0^{\sec^{-1}(\frac{5}{2})} \sec^3\theta - \sec\theta \cdot d\theta$ (we did $\int \sec^3\theta \, d\theta$ in previous example)

$= 4 \cdot \left[\frac{1}{2} \tan\theta \cdot \sec\theta + \frac{1}{2} \ln|\sec\theta + \tan\theta| - \ln|\sec\theta + \tan\theta| \right] \Big|_{\theta=0}^{\theta=\sec^{-1}(\frac{5}{2})}$

$= 4 \cdot \left[\frac{1}{2} \tan\theta \cdot \sec\theta - \frac{1}{2} \cdot \ln|\sec\theta + \tan\theta| \right] \Big|_{\theta=0}^{\theta=\sec^{-1}(\frac{5}{2})}$

[Note $\sec(\sec^{-1}(\frac{5}{2})) = \frac{5}{2}$

Now $\theta = \sec^{-1}(\frac{5}{2}) \Leftrightarrow \boxed{\sec\theta = \frac{5}{2}} = \frac{\text{hyp}}{\text{adj}}$



So $\boxed{\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{21}}{2}}$!]

and $\sec(0) = 1$, $\tan 0 = 0$

$= 4 \cdot \left\{ \left[\frac{1}{2} \cdot \frac{\sqrt{21}}{2} \cdot \frac{5}{2} - \frac{1}{2} \cdot \ln\left|\frac{5}{2} + \frac{\sqrt{21}}{2}\right| \right] - \cancel{\left[\frac{1}{2} \cdot 0 \cdot 1 - \frac{1}{2} \cdot \ln|1 + 0| \right]} \right\}$

$= 4 \cdot \left[\frac{1}{2} \cdot \frac{\sqrt{21}}{2} \cdot \frac{5}{2} - \frac{1}{2} \cdot \ln\left|\frac{5}{2} + \frac{\sqrt{21}}{2}\right| \right]$