Due in Class: April 23, 2015.

Turn in the following exercises. Exercise a.b refers to Exercise b in Chapter a in the Textbook.

Problem 1.

a. Let \( f(t) = t \) for \( t \in (-\pi, \pi] \), extended to be \( 2\pi \)-periodic on \( \mathbb{R} \). Compute the Fourier coefficients of \( f \).

b. Deduce that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).

Problem 2. Let \( u : C^\infty_c(\mathbb{R}^n) \to \mathbb{C} \) be a linear functional which is positive in the sense that \( \langle u, \phi \rangle \geq 0 \) whenever \( \phi \in C^\infty_c(\mathbb{R}^n) \) is positive. Show that \( u \) is a distribution of order zero.

Problem 3. Define \( u : C^\infty_c((0, \infty)) \to \mathbb{C} \) by

\[
\langle u, \phi \rangle := \sum_{n=1}^{\infty} \phi(n)(1/n) \quad (\phi \in C^\infty_c((0, \infty))).
\]

a. Show that \( u \) is a distribution on \((0, \infty)\) of infinite order.

b. Does there exist a distribution \( v \) on \( \mathbb{R} \) such that \( v = u \) on \((0, \infty)\)?

Problem 4. Let \( U \subset \mathbb{R}^n \) be open, let \( \{U_\alpha\}_{\alpha \in I} \) be an open cover of \( U \) and, for each \( \alpha \in I \), let \( u_\alpha \in \mathcal{D}'(U_\alpha) \). Suppose that \( u_\alpha = u_\beta \) on \( U_\alpha \cap U_\beta \), for all \( \alpha, \beta \in I \). Show that there is a unique \( u \in \mathcal{D}'(U) \) such that \( u = u_\alpha \) on \( U_\alpha \) for each \( \alpha \in I \). (Hint: Partition of unity.)

Problem 5.

a. For \( n \in \mathbb{N} \), define \( f_n : \mathbb{R} \to \mathbb{R} \) by

\[
f_n(x) := \int_{-n}^{nx} \frac{\sin t}{t} dt \quad (x \in \mathbb{R}).
\]

Show that \( f_n \to \pi H \) in \( \mathcal{D}'(\mathbb{R}) \) as \( n \to \infty \), where \( H = \chi_{(0,\infty)} \) is the Heaviside function. (Hint: You can take for granted the fact that \( \int_{-\infty}^{\infty} \frac{\sin t}{t} dt = \pi \).)

b. Deduce that \( \frac{\sin nx}{x} \to \pi \delta \) in \( \mathcal{D}'(\mathbb{R}) \) as \( n \to \infty \).
c. Deduce that if $\phi \in C^\infty_c(\mathbb{R})$, then
\[
\int_{-n}^{n} \left( \int_{-\infty}^{\infty} \phi(x)e^{-itx} \, dx \right) \, dt \to 2\pi \phi(0)
\]
as $n \to \infty$.

**Problem 6.** What is the order of the distribution p.v. $\frac{1}{x}$?

**Problem 7.** Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$. True or false?

a. If $\langle u, \phi \rangle = 0$, then $\phi u = 0$.

b. If $\phi u = 0$, then $\langle u, \phi \rangle = 0$. 