Problem 1. Show that the sum of two closed subsets of \( \mathbb{R}^n \) is not necessarily closed.

Sol. Let \( F_1 := \{(x, 0) : x \in \mathbb{R}\} \) and \( F_2 := \{(x, e^x) : x \in \mathbb{R}\} \). Then \( F_1 \) and \( F_2 \) are two closed subsets of \( \mathbb{R}^2 \), but \( F_1 + F_2 = \{(x, y) : x \in \mathbb{R}, y > 0\} \) is not closed.

Problem 2. Let \( A, B \) be two measurable subsets of \( \mathbb{R} \) with positive and finite measure. Show that \( A + B \) contains a segment. (Hint: Consider \( \chi_A \ast \chi_B \)).

Sol. First note that \( \chi_A \ast \chi_B \) is uniformly continuous on \( \mathbb{R} \), by a proposition proved in class. Furthermore, this function is not identically zero since by Fubini’s theorem and translation invariance of Lebesgue measure, we have

\[
\int \chi_A \ast \chi_B(x) \, dx = m(A)m(B) > 0.
\]

Let \( x_0 \in \mathbb{R} \) such that \( \chi_A \ast \chi_B(x_0) > 0 \), and let \( \delta > 0 \) such that \( |x - x_0| < \delta \) implies \( \chi_A \ast \chi_B(x) > 0 \). We claim that \( A + B \) contains the interval \( (x_0 - \delta, x_0 + \delta) \).

Problem 3. Exercise 8.15. (Note: In c., omit the part about uniform convergence).

a. Sol. A simple calculation shows that the Fourier transform of \( \chi_{[-a,a]} \) at \( \xi \in \mathbb{R} \) is \( 2a \text{sinc}(2a\xi) \).

b. Sol. Clearly \( H_a \) is a vector subspace of \( L^2 \). To show that it is a Hilbert space, it suffices to prove that it is closed in \( L^2 \). If \( (f_n) \subset H_a, f_n \to f \) in \( L^2 \), then \( \hat{f}_n \to \hat{f} \) in \( L^2 \) by Plancherel, hence there is a subsequence \( (f_{n_j}) \) such that \( \hat{f}_{n_j} \to \hat{f} \) almost everywhere. It easily follows that \( \hat{f}(\xi) = 0 \) for almost every \( |\xi| > a \), i.e. \( f \in H_a \). For the second part, set \( g_k(\xi) := \sqrt{2a} \text{sinc}(2a\xi - k) \) and \( h_a := \chi_{[-a,a]} \). Then for \( j, k \in \mathbb{Z} \), we have

\[
\langle g_j, g_k \rangle = \frac{1}{2a} \langle \hat{h}_a(\xi - j/2a), \hat{h}_a(\xi - k/2a) \rangle
\]

\[
= \frac{1}{2a} \langle (e^{2\pi i \frac{j}{2a} x} h_a)(\xi), (e^{2\pi i \frac{k}{2a} x} h_a)(\xi) \rangle
\]

\[
= \frac{1}{2a} \langle e^{2\pi i \frac{k}{2a} x} h_a, e^{2\pi i \frac{j}{2a} x} h_a \rangle = \delta_{jk},
\]
by Plancherel and the properties of the Fourier transform. This shows orthogonormality.

To prove that \( \{ g_k \}_{k \in \mathbb{Z}} \) is a basis, assume that \( g \in \mathcal{H}_a \) is orthogonal to every \( g_k \). Then by a calculation similar to the preceding one, we get that

\[
\int_{-a}^{a} \hat{g}(x)e^{i\pi k x/a} \, dx = 0 \quad (k \in \mathbb{Z})
\]
i.e.

\[
\int_{-\pi}^{\pi} \hat{g}(at/\pi)e^{ikt} \, dt = 0 \quad (k \in \mathbb{Z}).
\]
It follows from the density of \( \{ e^{ikt} : k \in \mathbb{Z} \} \) in \( L^2([\pi, \pi]) \), which is an easy consequence of Stone-Weierstrass, that \( \hat{g} = 0 \) a.e. on \([-a, a]\). Thus \( \hat{g} = 0 \) a.e. on \( \mathbb{R} \), since \( g \in \mathcal{H}_a \). Hence \( \hat{g} = 0 \) a.e. on \( \mathbb{R} \), since \( g \in \mathcal{H}_a \). Hence \( g = 0 \) a.e. and \( \{ g_k \}_{k \in \mathbb{Z}} \) is a basis.

c. Sol. If \( f \in \mathcal{H}_a \), then \( \hat{f} \in L^1 \) so by a Corollary of Plancherel’s theorem proved in class, we have that \( f(\xi) = \hat{f}(-\xi) \) almost everywhere, hence \( f \) is the Fourier transform of a function in \( L^1 \), which implies that \( f \in C_0 \) (Riemann-Lebesgue lemma). By part b., we get that

\[
f(x) = \sum_{k=-\infty}^{\infty} \langle f, g_k \rangle g_k(x)
\]
where the series converges in \( L^2 \). But for \( k \in \mathbb{Z} \), we have

\[
\langle f, g_k \rangle = \frac{1}{\sqrt{2a}} \langle f, \hat{h}_a(\xi - k/2a) \rangle
\]
\[
= \frac{1}{\sqrt{2a}} \langle \hat{f}(-\xi), (e^{2\pi i \frac{k}{2a} \xi}h_a)(\xi) \rangle
\]
\[
= \frac{1}{\sqrt{2a}} \langle \hat{f}(-\xi), e^{2\pi i \frac{k}{2a} \xi}h_a \rangle
\]
\[
= \frac{1}{\sqrt{2a}} \int_{-a}^{a} \hat{f}(-\xi)e^{-2\pi i \frac{k}{2a} \xi} \, d\xi
\]
\[
= \frac{1}{\sqrt{2a}} \hat{f} \left( -\frac{k}{2a} \right) = \frac{1}{\sqrt{2a}} f \left( \frac{k}{2a} \right),
\]
where we used Plancherel and the fact that \( \hat{f}(\xi) = 0 \) for all \( |\xi| > a \). Therefore, we obtain

\[
f(x) = \sum_{k=-\infty}^{\infty} f(k/2a) \text{sinc}(2ax - k),
\]
which is what we wanted to prove.

**Problem 4.** Let \( h := \chi_{[-1,1]} \) and for \( n \in \mathbb{N} \), let \( g_n := \chi_{[-n,n]} \).
a. Compute $g_n * h$ explicitly.

**Sol.** A simple calculation shows that

$$g_n * h(x) = \begin{cases} 
0 & \text{if } x \leq -n - 1 \\
x + 1 + n & \text{if } -n - 1 < x \leq -n + 1 \\
2 & \text{if } -n + 1 < x \leq n - 1 \\
n - x + 1 & \text{if } n - 1 < x \leq n + 1 \\
0 & \text{if } x > n + 1. 
\end{cases}$$

b. Show that $g_n * h$ is the Fourier transform of a function $f_n \in L^1(\mathbb{R})$ defined by

$$f_n(x) := \frac{\sin 2\pi x \sin 2\pi nx}{\pi^2 x^2}.$$

**Sol.** Note that $g_n * h \in L^1$ since $g_n, h \in L^1$. By the inversion theorem, $g_n * h$ is the Fourier transform of $f_n(x) := (\hat{g}_n \hat{h})(-x)$. A simple calculation using Problem 3 shows that $f_n$ has the desired form.

c. Show that $\|f_n\|_1 \to \infty$ as $n \to \infty$.

**Sol.** Setting $y = 2\pi nx$ in the integral for $\|f_n\|_1$, we get

$$\|f_n\|_1 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{|\sin y/n| \sin y}{|y/n||y|} dy$$

hence by Fatou’s lemma,

$$\liminf_{n \to \infty} \|f_n\|_1 \geq \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{|\sin y|}{|y|} dy = \infty.$$

d. Deduce that $f \mapsto \hat{f}$ maps $L^1(\mathbb{R})$ onto a proper subset of $C_0(\mathbb{R})$.

**Sol.** Suppose for a contradiction that $f \mapsto \hat{f}$ maps $L^1(\mathbb{R})$ onto $C_0(\mathbb{R})$. Since $\|\hat{f}\|_u \leq \|f\|_1$ for all $f \in L^1$ and the Fourier transform is one-to-one on $L^1$ by the inversion theorem, this would mean that $f \mapsto \hat{f}$ is a bounded linear bijection between the Banach spaces $L^1(\mathbb{R})$ and $C_0(\mathbb{R})$. By the open mapping theorem, there is a constant $C > 0$ such that

$$\|\hat{f}\|_u \geq C\|f\|_1 \quad (f \in L^1(\mathbb{R})).$$

However, note that the functions $f_n$ as above belong to $L^1(\mathbb{R})$ and satisfy $\|\hat{f}_n\|_u = \|g_n * h\|_u = 2$ for all $n$, yet $\|f_n\|_1 \to \infty$ as $n \to \infty$. This is a contradiction.

**Problem 5.** Use Exercise 8.15a to deduce the Fourier transform of $(\sin x/x)^2$. 
Let \( f(x) = (\sin 2\pi x/x)^2 \) and let \( h = \chi_{[-1,1]} \). By Problem 3, \( \hat{h}(\xi) = \sin 2\pi \xi/(\pi \xi) \), so that
\[
\hat{f}(\xi) = \pi^2(\hat{h}\hat{h})(\xi) = \pi^2(h \ast h)(\xi) = \pi^2(h \ast h)(-\xi).
\]
Let \( g(x) = (\sin x/x)^2 = (2\pi)^{-2}f(x/2\pi) \). Then
\[
\hat{g}(\xi) = (2\pi)^{-1}\hat{f}(2\pi\xi) = \frac{\pi}{2}(h \ast h)(-2\pi\xi).
\]
Problem 4 part a then gives an explicit expression for \( \hat{g}(\xi) \).

Problem 6.
a. Compute the Fourier transform of \( e^{-|x|} \) on \( \mathbb{R} \).

Sol. A simple calculation shows that the Fourier transform of \( f(x) := e^{-|x|} \) is \( f(\xi) = 2/(1 + 4\pi^2\xi^2) \).

b. Deduce the value of the integral
\[
\int_{-\infty}^{\infty} dx (1 + x^2)^{-2}.
\]

Sol. By a simple change of variable and Pancherel, we get
\[
\int_{-\infty}^{\infty} dx (1 + x^2)^{-2} = \frac{\pi}{2} \int_{-\infty}^{\infty} \left| \hat{f}(\xi) \right|^2 d\xi = \frac{\pi}{2} \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{\pi}{2}.
\]

Problem 7. Does there exist a function \( u \in L^1(\mathbb{R}^n) \) such that \( f \ast u = f \) for all \( f \in L^1(\mathbb{R}^n) \)?

Sol. If such a function \( u \in L^1(\mathbb{R}^n) \) exists, then we must have \( \hat{f} \hat{u} = \hat{f} \) for all \( f \in L^1(\mathbb{R}^n) \). Apply this to some \( f \in L^1(\mathbb{R}^n) \) such that \( \hat{f} \) is nowhere vanishing, for instance \( f(x) = e^{-|x|^2} \). We obtain that \( \hat{u} \) is identically equal to one, which contradicts the fact that \( \hat{u} \in C_0(\mathbb{R}^n) \).

Problem 8.
a. Let \( T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n) \) be a linear map such that
\[
T(\partial_j \phi) = \partial_j T(\phi) \quad \text{and} \quad T(x_j \phi) = x_j T(\phi) \quad (\phi \in \mathcal{S}(\mathbb{R}^n), j = 1, \ldots, n).
\]
Show that \( T \) is a multiple of the identity.

(Hint : You can take for granted the fact that if \( \phi \in \mathcal{S}(\mathbb{R}^n) \) and \( y \in \mathbb{R}^n \) is such that \( \phi(y) = 0 \), then there exist \( \phi_1, \ldots, \phi_n \in \mathcal{S}(\mathbb{R}^n) \) such that \( \phi(x) = \sum_{j=1}^{n}(x_j - y_j)\phi_j(x) \) for all \( x \in \mathbb{R}^n \).)

Sol. First we show that if \( \phi \in \mathcal{S} \) and \( y \in \mathbb{R}^n \) is such that \( \phi(y) = 0 \), then \( T\phi(y) = 0 \). By the hint, there exist \( \phi_1, \ldots, \phi_n \in \mathcal{S}(\mathbb{R}^n) \) such that
\[ \phi(x) = \sum_{j=1}^{n}(x_j - y_j)\phi_j(x) \text{ for all } x \in \mathbb{R}^n. \] Thus

\[ T\phi(x) = \sum_{j=1}^{n}(x_j - y_j)T\phi_j(x) \quad (x \in \mathbb{R}^n) \]

so \( T\phi(y) = 0 \). Now, fix \( x_0 \in \mathbb{R}^n \) and let \( \phi \in S \). Let \( \psi \in C_c^\infty(\mathbb{R}^n) \) with \( \psi(x_0) = 1 \). Then we have

\[ T\phi(x_0) = T(\phi - \phi(x_0)\psi)(x_0) + T(\phi(x_0)\psi)(x_0) = \phi(x_0)T(\psi)(x_0), \]

since the function \( \phi - \phi(x_0)\psi \) vanishes at \( x_0 \). But \( x_0 \) was arbitrary, so \( T(\phi) = \phi f \) for some function \( f \) on \( \mathbb{R}^n \). Applying this to a function \( \phi \in S \) which is nonvanishing on a given ball shows that \( f \) is \( C^\infty \) on that ball, hence \( f \in C^\infty(\mathbb{R}^n) \). Finally, fix \( j \in \{1, \ldots, n\} \). Since \( T \) commutes with \( \partial_j \), we get

\[ f(\partial_j \phi) = \partial_j(f\phi) = \phi\partial_j f + f\partial_j \phi \quad (\phi \in S), \]

hence \( \phi\partial_j f \equiv 0 \) for all \( \phi \in S \), which implies that \( \partial_j f \equiv 0 \). Since this holds for each \( j \), we get that \( f \) is constant. Therefore, \( T\phi = c\phi \) is a multiple of the identity.

b. Use part a. to give another proof of the Fourier Inversion Theorem.

**Sol.** Let \( T : S \to S \) defined by \( T\phi(x) = \hat{\phi}(-x) \). Simple manipulations show that \( T \) satisfies the hypotheses of a., so \( T\phi = c\phi \) for some constant \( c \). Applying \( T \) to \( \phi(x) = e^{-\pi|x|^2} \), we get that \( c = 1 \).

**Problem 9.** Prove that if \( \phi \) is a complex homomorphism on a Banach algebra \( A \), then \( \phi \) is a bounded linear functional of norm at most one.

**Sol.** Assume, to get a contradiction, that \( |\phi(x_0)| > \|x_0\| \) for some \( x_0 \in A \). Put \( \lambda = \phi(x_0) \) and set \( x = x_0/\lambda \). Then \( \|x\| < 1 \) and \( \phi(x) = 1 \). Since \( \|x^n\| \leq \|x\|^n \) and \( \|x\| < 1 \), the elements

\[ s_n = -x - x^2 - \cdots - x^n \]

form a Cauchy sequence in \( A \). By completeness, there exists \( y \in A \) such that \( s_n \to y \) in \( A \). Clearly \( x + s_n = xs_{n-1} \) for all \( n \), so that

\[ x + y = xy \]

and thus

\[ 1 + \phi(y) = \phi(x) + \phi(y) = \phi(x + y) = \phi(xy) = \phi(x)\phi(y) = \phi(y), \]

a contradiction.

**Problem 10.** Is \( L^2(\mathbb{R}) \) closed under convolution?

**First Sol.** Suppose that \( L^2(\mathbb{R}) \) is closed under convolution. First, we prove that this implies that there is a constant \( C > 0 \) such that

\[ \|f * g\|_2 \leq C\|f\|_2\|g\|_2 \quad (f, g \in L^2(\mathbb{R})). \]
For \( g \in L^2(\mathbb{R}) \), define \( T_g : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) by
\[
T_g(f) = f \ast g \quad (f \in L^2(\mathbb{R})).
\]
Clearly, for each \( g \), \( T_g \) is a linear operator. Let us show that \( T_g \) is bounded with the Closed Graph Theorem. Assume that \( f_n \to f \) in \( L^2(\mathbb{R}) \) and \( T_g(f_n) = f_n \ast g \to h \) in \( L^2(\mathbb{R}) \). Then we have
\[
\|f_n \ast g - f \ast g\|_u \leq \|f_n - f\|_2 \|g\|_2 \to 0
\]
as \( n \to \infty \). Taking a subsequence such that \( f_{n_j} \ast g \to h \) almost everywhere shows that \( T_g(f) = f \ast g = h \) almost everywhere. This proves that each \( T_g \) is bounded, i.e. \( \|f \ast g\|_2 \leq \|T_g\| \|f\|_2 \) for all \( f \in L^2(\mathbb{R}) \). Consider now the family of bounded linear operators \( \{T_g\} \) where \( g \in L^2(\mathbb{R}), \|g\|_2 \leq 1 \). Fix \( f \in L^2(\mathbb{R}) \). Then
\[
\|T_g(f)\| \leq \|T_g\| \|g\|_2 \leq \|T_g\| \quad (g \in L^2(\mathbb{R}), \|g\|_2 \leq 1).
\]
Thus we are in the first case of the Uniform Boundedness Principle, so that \( \|T_g\| \leq C < \infty \) for all \( g \in L^2(\mathbb{R}), \|g\|_2 \leq 1 \). Normalizing, we get
\[
\|f \ast g\|_2 \leq C \|f\|_2 \|g\|_2 \quad (f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R})).
\]

Now, let \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), f \neq 0 \) and for \( t > 0 \), define \( f_t(x) = f(x/t) \). By the above inequality with \( f = g = f_t \) and using the fact that \( (f_t \ast f_t)^\sim = f_t^2 \), we get
\[
\|f_t^2\|_2 \leq C \|f_t\|_2^2.
\]
But \( \hat{f}_t(\xi) = t\hat{f}(t\xi) \), so a simple change of variable yields
\[
\sqrt{t} \|f^2\|_2 \leq C \|f\|_2^2.
\]
Letting \( t \to \infty \) gives a contradiction. 

**Second Sol.** First, we show that if \( f, g \in L^2(\mathbb{R}) \) and \( f \ast g \in L^2(\mathbb{R}) \), then \( (f \ast g)^\sim = \hat{f} \hat{g} \). To prove this, first note that \( \hat{f} \hat{g} \in L^1(\mathbb{R}) \), so its Fourier transform at \( \xi \in \mathbb{R} \) is
\[
\int \hat{f}(x) \hat{g}(x) e^{-2\pi i \xi x} \, dx = \langle f, \overline{\hat{g}} e^{2\pi i \xi x} \rangle = \langle f, \overline{\hat{g}} e^{2\pi i \xi x} \rangle = \langle f, (\tau_{-\xi} \hat{g})^\sim \rangle = \langle f, \tau_{-\xi} \overline{\hat{g}} \rangle = \int f(x) g(-\xi - x) \, dx = (f \ast g)(-\xi),
\]
where \( \overline{\hat{g}}(x) := g(-x) \). We used Plancherel and also the fact that \( \overline{\hat{g}} = \overline{\hat{g}} \). In particular, the Fourier transform of \( \hat{f} \hat{g} \) belongs to \( L^2(\mathbb{R}) \), which implies
that \( \hat{f} \hat{g} \in L^2(\mathbb{R}) \), by Plancherel. Let \( h \in L^2(\mathbb{R}) \) be such that \( \hat{h} = \hat{f} \hat{g} \). Since \( \hat{f} \hat{g} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), we have, by a Corollary proved in class and the above calculation,

\[
h(x) = \int \hat{f}(\xi)\hat{g}(\xi) e^{2\pi i \xi x} \, d\xi = (f \ast g)(x) \quad (a.e. \, x \in \mathbb{R}),
\]

and thus \( \hat{f} \hat{g} = \hat{h} = (f \ast g)\). 

Now, consider a function \( F \in L^2(\mathbb{R}) \) such that \( F^2 \notin L^2(\mathbb{R}) \), for instance \( F(x) := x^{-1/3}e^{-x^2} \). Let \( f \in L^2(\mathbb{R}) \) such that \( \hat{f} = F \). We claim that \( f \ast f \notin L^2(\mathbb{R}) \). Indeed, if \( f \ast f \in L^2(\mathbb{R}) \), then by the above formula with \( g = f \), we get

\[
F^2 = \hat{f} \hat{f} = (f \ast f)^\ast \in L^2(\mathbb{R}),
\]

a contradiction.