Due in Class: April 16, 2015.

Turn in the following exercises. Exercise a.b refers to Exercise b in Chapter a in the Textbook.

Problem 1. Show that the sum of two closed subsets of \( \mathbb{R}^n \) is not necessarily closed.

Problem 2. Let \( A, B \) be two measurable subsets of \( \mathbb{R} \) with positive and finite measure. Show that \( A + B \) contains a segment.
(Hint: Consider \( \chi_A \ast \chi_B \)).

Problem 3. Exercise 8.15. (Note: In c., omit the part about uniform convergence).

Problem 4. Let \( h := \chi_{[-1,1]} \) and for \( n \in \mathbb{N} \), let \( g_n := \chi_{[-n,n]} \).

a. Compute \( g_n \ast h \) explicitly.

b. Show that \( g_n \ast h \) is the Fourier transform of a function \( f_n \in L^1(\mathbb{R}) \) defined by
\[
f_n(x) := \frac{\sin 2\pi x \sin 2\pi nx}{\pi^2 x^2}.
\]

C. Show that \( \| f_n \|_1 \to \infty \) as \( n \to \infty \).

d. Deduce that \( f \mapsto \hat{f} \) maps \( L^1(\mathbb{R}) \) onto a proper subset of \( C_0(\mathbb{R}) \).

Problem 5. Use Exercise 8.15a to deduce the Fourier transform of \((\sin x/x)^2\).

Problem 6.

a. Compute the Fourier transform of \( e^{-|x|} \) on \( \mathbb{R} \).

b. Deduce the value of the integral
\[
\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2}.
\]

Problem 7. Does there exist a function \( u \in L^1(\mathbb{R}^n) \) such that \( f \ast u = f \) for all \( f \in L^1(\mathbb{R}^n) \)?

Problem 8.

a. Let \( T : S(\mathbb{R}^n) \to S(\mathbb{R}^n) \) be a linear map such that
\[
T(\partial_j \phi) = \partial_j T(\phi) \quad \text{and} \quad T(x_j \phi) = x_j T(\phi) \quad (\phi \in S(\mathbb{R}^n), j = 1, \ldots, n).
\]

Show that \( T \) is a multiple of the identity.
(Hint: You can take for granted the fact that if \( \phi \in \mathcal{S}(\mathbb{R}^n) \) and \( y \in \mathbb{R}^n \) is such that \( \phi(y) = 0 \), then there exist \( \phi_1, \ldots, \phi_n \in \mathcal{S}(\mathbb{R}^n) \) such that \( \phi(x) = \sum_{j=1}^{n} (x_j - y_j)\phi_j(x) \) for all \( x \in \mathbb{R}^n \)).

b. Use part a. to give another proof of the Fourier Inversion Theorem.

**Problem 9.** Prove that if \( \phi \) is a complex homomorphism on a Banach algebra \( A \), then \( \phi \) is a bounded linear functional of norm at most one.

**Problem 10.** Is \( L^2(\mathbb{R}) \) closed under convolution?