Problem 1. Show that a subset $E$ of $\mathbb{R}$ is the support of a continuous function $f : \mathbb{R} \to \mathbb{C}$ if and only if it is the closure of an open set.

If $E \subset \mathbb{R}$ is the support of a continuous function $f : \mathbb{R} \to \mathbb{C}$, then $E$ is the closure of the open set $\{x \in \mathbb{R} : f(x) \neq 0\}$.

Conversely, assume that $E = \overline{V}$ for some open set $V \subset \mathbb{R}$. If $E$ is empty, then $E$ is the support of the zero function. Otherwise, write $V$ as the countable union of disjoint open intervals $(a_j, b_j)$. For each $j$, let $f_j : \mathbb{R} \to \mathbb{C}$ be a positive continuous function which is nonzero on $(a_j, b_j)$ and vanishes identically on the complement of $(a_j, b_j)$. For instance, one can take $f_j(x) := \text{dist}(x, (a_j, b_j))$. Define $f : \mathbb{R} \to \mathbb{C}$ by

$$f(x) = \sum_{j=1}^{\infty} \frac{f_j(x)}{2^j \|f_j\|_u}.$$ 

Then the series converges uniformly by the Weierstrass M-test, hence $f$ is continuous. Finally, it is easy to see that $V = \{x \in \mathbb{R} : f(x) \neq 0\}$. Therefore, $E$ is the support of $f$.

Problem 2. Exercise 7.2.

a. Let $N$ be the union of all open sets $U \subset X$ such that $\mu(U) = 0$. Then $N$ is open since it is a union of open sets. If $K$ is any compact subset of $N$, then by compactness and the definition of $N$ we get that $K$ is contained in a finite union of open sets of measure zero, so $\mu(K) = 0$. By inner regularity of $\mu$ on $N$, it follows that $\mu(N) = 0$.

b. If $x \in N$, then there is an open set $U$ with $x \in U$ and $\mu(U) = 0$. Let $f \in C_c(X)$ such that $\{x\} \prec f \prec U$. Then $0 \leq f \leq 1$, $f(x) = 1 > 0$ and $\int f \, d\mu \leq \int_U d\mu = 0$.

Conversely, assume that $x \in \text{supp}(\mu)$. Let $f \in C_c(X)$, $0 \leq f \leq 1$ with $f(x) > 0$. Then $V := \{f > 0\}$ is open and contains $x$. Since $x$ belongs to the support of $\mu$, we must have $\mu(V) > 0$. It follows that $\int f \, d\mu \geq \int_V f \, d\mu > 0$.

Problem 3. Let $X$ be a locally compact Hausdorff topological space and let $\mu$ be a positive regular measure on $X$. Suppose that $f : X \to \mathbb{C}$ is measurable and vanishes outside a set of finite measure. Show that if $|f| \leq 1$, then there is a sequence $\{g_n\}_{n \geq 1}$ such that $g_n \in C_c(X)$, $|g_n| \leq 1$ for all $n$ and

$$f(x) = \lim_{n \to \infty} g_n(x) \quad \text{a.e.}$$
By Lusin’s theorem, for each $n$, there is a function $g_n \in C_c(X)$ with $|g_n| \leq 1$ such that $\mu(E_n) \leq 2^{-n}$, where

$$E_n := \{x \in X : f(x) \neq g_n(x)\}.$$  

This implies that for almost every $x \in X$, we have $f(x) = g_n(x)$ for all $n$ sufficiently large. Indeed, let $A$ be the set of all $x \in X$ which lie in infinitely many $E_n$’s. Put

$$g(x) := \sum_{n=1}^{\infty} \chi_{E_n}(x) \quad (x \in X).$$

Then clearly $g(x) = \infty$ if and only if $x \in A$. By the monotone convergence theorem, we have

$$\int_X |g| \, d\mu = \sum_{n=1}^{\infty} \mu(E_n) \leq \sum_{n=1}^{\infty} 2^{-n} < \infty,$$

so that $g \in L^1(\mu)$ and therefore $g(x) < \infty$ almost everywhere. In other words, $\mu(A) = 0$. It follows that

$$f(x) = \lim_{n \to \infty} g_n(x) \quad a.e.$$