Problem 1. Exercise 5.2.

Denote by $\alpha_1, \alpha_2, \alpha_3$ the three expressions of (5.3). Let us first prove that $\alpha_1 = \alpha_2 = \alpha_3$. If $x \in X, x \neq 0$, then $x/\|x\|$ has norm 1 and thus $T(x)/\|x\| = T(x)/\|x\| \leq \alpha_1$. Taking the supremum shows that $\alpha_2 \leq \alpha_1$. If $x \in X, \|x\| = 1$, then $\|Tx\| \leq \alpha_3$ and therefore $\alpha_3 \leq \alpha_2$. Finally, if $x \in X, \|x\| = 1$, then $\|Tx\| \leq \alpha_3$ and taking the supremum over such $x$’s yield $\alpha_1 \leq \alpha_3$.

Let us now prove that $\|\cdot\|$ defines a norm on $L(X,Y)$. If $S, T \in L(X,Y)$ and $x \in X$, then

$$\|(S + T)x\| = \|Sx + Tx\| \leq \|Sx\| + \|Tx\| \leq (\|S\| + \|T\|)\|x\|$$

and hence $\|S + T\| \leq \|S\| + \|T\|$. If $\lambda \in \mathbb{C}$, then $\|\lambda Tx\| = |\lambda|\|Tx\| \leq |\lambda|\|T\||\|x\|$, which shows that $\|\lambda T\| \leq |\lambda|\|T\|$. For the reverse inequality, replace $\lambda$ by $\lambda^{-1}$ and $T$ by $\lambda T$. Lastly, if $\|T\| = 0$ then $\|Tx\| = 0$ for all $x$ and $T = 0$.

Problem 2. Exercise 5.4.

Suppose that $T_n \to T$ and $x_n \to x$. Then we have

$$\|T_n x_n - T x\| = \|T_n(x_n - x) + (T_n - T)x\| \leq \|T_n\|\|x_n - x\| + \|T_n - T\|\|x\|$$

and the right-hand side tends to 0 as $n \to \infty$.

Problem 3. Exercise 5.7.

a. Note that if $S$ is any bounded linear operator and $n \in \mathbb{N}$, then $\|S^n\| \leq \|S\|^n$. Hence $\|(I - T)^n\| \leq \|(I - T)\|^n$ and since $\|I - T\| < 1$, we have that the partial sums of $\sum (I - T)^n$ form a Cauchy sequence in $L(X,X)$, which converges to some element $S$ by completeness. If $S_N$ denotes the $N$-th partial sum, then we have $TS_N = S_N T = I - (I - T)^{N+1}$ and letting $N \to \infty$ shows that $S = T^{-1}$.

b. Note that $\|I - T^{-1} S\| = \|T^{-1}(T - S)\| \leq \|T^{-1}\|\|T - S\| < 1$, so by a. the operator $T^{-1} S$ is invertible. Clearly, this implies that $S$ is invertible.

Problem 4. Exercise 5.25.

We follow the suggestion and let $x_n \in X$ such that $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$, where $(f_n)$ is a countable dense subset of $X^*$. We claim that span$\{x_n\}$
is dense in $X$. This implies separability of $X$, since one can obtain a countable dense subset by considering linear combinations of the $x_n$'s with coefficients in $\mathbb{Q} + i\mathbb{Q}$. To prove the claim, suppose the contrary and let $M := \text{span}\{x_n\}$, so that $M \neq X$. By the first consequence of the Hahn-Banach theorem, there is a nonzero $f \in X^*$ which vanishes on $M$. Let $(f_{n_j})$ be a subsequence of the dense sequence $(f_n)$ with $f_{n_j} \to f$. Then we have

$$\frac{1}{2}\|f_{n_j}\| \leq |f_{n_j}(x_{n_j}) - f(x_{n_j})| \leq \|f_{n_j} - f\|.$$ 

Letting $j \to \infty$ gives $\|f\| = 0$, a contradiction.

**Problem 5.** Exercise 5.27. Note: The term *Meager* means that the set is a countable union of nowhere dense sets.

Let $\{r_j\}$ be an enumeration of the rationals and for $j, n \in \mathbb{N}$, let

$$V_{j,n} := \left\{ x \in \mathbb{R} : |x - r_j| < \frac{1}{n2^{j+1}} \right\}.$$ 

Define $A := \bigcup_n \cap_j V_{j,n}$. Then $m(A^c) \leq 1/n$ for all $n$, so $m(A^c) = 0$. Moreover, it is easy to see that for each $n$, $\cap_j V_{j,n}$ has empty interior. Therefore $A$ is meagre.

**Problem 6.** Exercise 5.32.

It suffices to apply the Corollary to the Open Mapping Theorem to the identity map from $(X, \| \cdot \|_2)$ to $(X, \| \cdot \|_1)$.

**Problem 7.** Exercise 5.38.

Clearly $T$ is linear. Since $\sup_n \|T_n x\| < \infty$ for all $x \in X$, we are in the first case of the Uniform Boundedness Principle and $M := \sup_n \|T_n\| < \infty$. For all $n$ and all $x$, we have $\|T_n x\| \leq \|T_n\| \|x\| \leq M \|x\|$. Letting $n \to \infty$ shows that $\|T x\| \leq M \|x\|$ for all $x$, so that $T$ is bounded and $\|T\| \leq M$.

**Problem 8.** Show that if $(\alpha_j)$ is a sequence of complex numbers such that $\sum_j \alpha_j \xi_j$ converges for every sequence $(\xi_j)$ of complex numbers with $\xi_j \to 0$ as $j \to \infty$, then $\sum_j |\alpha_j| < \infty$.

Let $c_0$ denote the space of all sequences of complex numbers which tend to zero, equipped with the uniform norm

$$\|x\| := \sup_j |\xi_j| \quad (x = (\xi_1, \xi_2, \ldots)).$$

It is easy to see that $c_0$ is a Banach space (it is closed in the set of all bounded sequences). For $n \in \mathbb{N}$ and $x = (\xi_1, \xi_2, \ldots) \in c_0$, let $T_n(x) := \sum_{j=1}^n \alpha_j \xi_j$. Clearly $T_n$ is linear. Also, $|T_n(x)| \leq (\sum_{j=1}^n |\alpha_j|) \|x\|$, so that $T_n$ is bounded.
and $\|T_n\| \leq \sum_{j=1}^{n} |\alpha_j|$. By taking $x \in c_0$ defined by $\xi_j = \alpha_j/|\alpha_j|$ if $1 \leq j \leq n$ and $\alpha_j \neq 0$ and zero otherwise, we see that in fact $\|T_n\| = \sum_{j=1}^{n} |\alpha_j|$. Since $\lim_{n \to \infty} T_n(x) \text{ converges for every } x \in c_0$, we have that $\sup_n \|T_n\| < \infty$ by the Uniform Boundedness Principle. In other words, $\sum_{j=1}^{\infty} |\alpha_j| < \infty$.

**Problem 9.** Does there exist a sequence of continuous positive functions $f_n$ on $\mathbb{R}$ such that the sequence $(f_n(x))$ is unbounded if and only if $x$ is rational? *Hint: Is $\mathbb{Q}$ a $G_\delta$?*

No, such a sequence does not exist. First note that $\mathbb{Q}$ is not a $G_\delta$. Indeed, if $\mathbb{Q} := \{r_n\}$ were a $G_\delta$, say $\mathbb{Q} = \cap_n V_n$ where each $V_n$ is open, then

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \{r_n\} \cup (\cup_n V_n^c)$$

would be a countable union of nowhere dense sets, contradicting the Baire Category theorem. However, the set of points $A$ at which a sequence of positive continuous functions is unbounded is a $G_\delta$ :

$$A = \cap_n \cup_n \{x \in \mathbb{R} : f_n(x) > m\}.$$