Problem 1. Let $V \subset \mathbb{R}^n$ be open and let $u \in \mathcal{D}'(\mathbb{R} \times V)$. Show that $x_1 u = 0$ if and only if $u = \delta \otimes v$, where $v \in \mathcal{D}'(V)$ and $\delta$ is the Dirac distribution on $\mathbb{R}$.

Sol. If $u = \delta \otimes v$ for some $v \in \mathcal{D}'(V)$, then clearly $x_1 u = (x_1 \delta) \otimes v = 0 \otimes v = 0$. Conversely, assume that $x_1 u = 0$. For $\chi \in C_c^\infty(V)$, define $v_\chi : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$\langle v_\chi, \phi \rangle = \langle u, \phi \otimes \chi \rangle \quad (\phi \in C_c^\infty(\mathbb{R})).$$

Clearly, $v_\chi \in \mathcal{D}'(\mathbb{R})$. Moreover, we have

$$\langle x_1 v_\chi, \phi \rangle = \langle u, (x_1 \phi) \otimes \chi \rangle = \langle x_1 u, \phi \otimes \chi \rangle = 0$$

since $x_1 u = 0$. Thus $v_\chi = C_\chi \delta$ for some $C_\chi \in \mathbb{C}$, by the solution to the division problem. Define $v : C_c^\infty(V) \to \mathbb{C}$ by $\langle v, \chi \rangle := C_\chi$. Then for $\phi \in C_c^\infty(\mathbb{R})$ and $\chi \in C_c^\infty(V)$, we have

$$\langle u, \phi \otimes \chi \rangle = \langle v_\chi, \phi \rangle = \langle \delta, \phi \rangle \langle v, \chi \rangle.$$

Taking $\phi_0 \in C_c^\infty(\mathbb{R})$ with $\phi_0(0) = 1$, we get $\langle v, \chi \rangle = \langle u, \phi_0 \otimes \chi \rangle$, which shows that $v \in \mathcal{D}'(V)$. Finally, the above equality shows that $u = \delta \otimes v$.

Problem 2.

a. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and define $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by $T \phi := u \ast \phi$. Show that

1. $\tau_h(T \phi) = T(\tau_h \phi)$ for all $h \in \mathbb{R}^n$ and all $\phi \in C_c^\infty(\mathbb{R}^n)$,
2. $(T \phi_n)(0) \to (T \phi)(0)$ whenever $\phi_n \to \phi$ in $C_c^\infty(\mathbb{R}^n)$.

Sol. Both statements follow directly from the regularization theorem.

b. Conversely, show that if $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a linear map satisfying (1) and (2), then there exists $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $T \phi = u \ast \phi$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

Sol. Define $u : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ by

$$\langle u, \phi \rangle := T \phi(0) \quad (\phi \in C_c^\infty(\mathbb{R}^n)).$$

Then $u$ is linear and sequentially continuous by (2). Moreover, for $\phi \in C_c^\infty(\mathbb{R}^n)$, we have, by the regularization theorem

$$(u \ast \phi)(x) = \langle u(y), \phi(x - y) \rangle = T(\tau_x \phi)(0) = \tau_x(T \phi)(0) = T \phi(x) \quad (x \in \mathbb{R}^n),$$

by (1). Hence $T \phi = u \ast \phi$.

Problem 3.
a. Show that 
\[ E(z) := \frac{1}{\pi z} \]
is a fundamental solution of the Cauchy-Riemann operator \( \partial/\partial \bar{z} \) on \( \mathbb{C} = \mathbb{R}^2 \).

**Sol.** Proceeding as in the example of the Laplacian, we get that \( \partial E/\partial \bar{z} = c_0 \delta \) for some \( c_0 \in \mathbb{C} \). To find the value of \( c_0 \), fix \( \phi \in C^\infty_c (\mathbb{C}) \) radial. Then \( \partial \phi / \partial \bar{z} = (e^{i\theta}/2) \partial \phi / \partial r \), so that 
\[ \left\langle \frac{\partial E}{\partial \bar{z}}, \phi \right\rangle = -\lim_{\epsilon \to 0} \int_0^\infty \int_0^{2\pi} \frac{1}{\pi r e^{i\theta}} \frac{e^{i\theta}}{2} \frac{\partial \phi}{\partial r} r d\theta dr = \phi(0). \]
Thus \( c_0 = 1 \) and \( \partial E/\partial \bar{z} = \delta \).

b. Find a fundamental solution of the differential operator \( \partial^\alpha \) on \( \mathbb{R}^n \).

**Sol.** Write \( \alpha = (\alpha_1, \ldots, \alpha_n) \). Let us first find a fundamental solution \( u_j \) of the differential operator \( \partial_{x_j}^{\alpha_j} \) on \( \mathbb{R} \). If \( \alpha_j = 0 \), then \( u_j = \delta \) is trivially a fundamental solution. If \( \alpha_j \geq 1 \), then a simple calculation shows that 
\[ \partial_{x_j}^{\alpha_j} (x_j^{\alpha_j-1} H_{x_j}) = (\alpha_j - 1)! \delta, \]
so that \( u_j := 1/(\alpha_j - 1)! x_j^{\alpha_j-1} H_{x_j} \) is a fundamental solution. Now, for a fundamental solution of \( \partial^\alpha \) on \( \mathbb{R}^n \), it suffices to consider \( u := u_1 \otimes u_2 \otimes \cdots \otimes u_n \).

**Problem 4.** Show that \( e^x \cos(e^x) \) is a tempered distribution on \( \mathbb{R} \), while \( e^x \) is not.

**Sol.** Since \( e^x \cos(e^x) = (\sin(e^x))' \) and \( \sin(e^x) \in L^\infty(\mathbb{R}) \subset S'(\mathbb{R}) \), it follows that \( e^x \cos(e^x) \in S'(\mathbb{R}) \).

For the second part, set \( u := e^x \) and assume that \( u \in S'(\mathbb{R}) \). Since \( u' - u = 0 \), we can take the Fourier transform on both sides to deduce that \( (2\pi i \xi - 1)u' = 0 \). Since \( 2\pi i \xi - 1 \neq 0 \) for all \( \xi \in \mathbb{R} \), it easily follows that \( u' = 0 \), so that \( u = 0 \), which is a contradiction.

**Problem 5.** Let \( u \in S'(\mathbb{R}^n) \) be a tempered distribution that is homogeneous of degree \( \lambda \). Show that its Fourier transform \( \hat{u} \) is also homogeneous, and find its degree.

**Sol.** For \( t > 0 \), let \( f(x) = tx \) and \( u_t := u \circ f \). Then \( t^{-\lambda} \hat{u} = (u_t) \) and \( \hat{u} = t^{-\lambda-n} \hat{u} \) and \( \hat{u} \) is homogeneous of degree \( -\lambda - n \).

**Problem 6.** Find all distributions \( u \in S'(\mathbb{R}^n) \) such that \( \Delta u = u \). Is the answer different if we replace \( S' \) by \( \mathcal{D}' \)?
Sol. If $u \in S'(\mathbb{R}^n)$ and $\Delta u = u$, then by taking the Fourier transform we get $-4\pi^2|\xi|^2\hat{u} = \hat{u}$, i.e. $(1 + 4\pi^2|\xi|^2)\hat{u} = 0$. It easily follows that $\hat{u} = 0$ and so $u = 0$.

The answer is different if we replace $S'$ by $D'$. Indeed, $u = e^x$ on $\mathbb{R}$ satisfies $\Delta u = u$. 