Problem 1.

Prove that there does not exist a rational number whose square is 10.

Sol. Assume for a contradiction that $\sqrt{10} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. We can assume that the fraction $a/b$ is in lowest terms. Then we have

$$10b^2 = a^2$$

which implies that $a^2$, and therefore $a$, is even. Write $a = 2a_0$ for some $a_0 \in \mathbb{Z}$. Then we have

$$10b^2 = 4a_0^2$$

i.e.

$$5b^2 = 2a_0^2$$

and so $5b^2$ is even. It is easy to see that this implies that $b$ is even, a contradiction since we assumed that $a$ and $b$ had no common factor.

Problem 2.

Prove that the set of polynomials with integer coefficients is denumerable. Deduce that the set of algebraic numbers is denumerable.

Hint : You can take for granted the fact that a polynomial equation

$$a_0 + a_1 x + \ldots + a_n x^n = 0$$

has at most $n$ real solutions.

Sol. For $n \in \mathbb{Z}_{\geq 1}$, let $P_n$ be the set of all polynomials of degree $n$ with integer coefficients. Define a function $f : P_n \to \mathbb{Z}^{n+1}$ by

$$a_0 + a_1 x + \cdots + a_n x^n \mapsto (a_0, a_1, \ldots, a_n).$$

Clearly, $f$ is bijective, so that $P_n$ is denumerable, since $\mathbb{Z}^{n+1}$ is denumerable. Now, note that the set of polynomials with integer coefficients is the union over all $n \in \mathbb{Z}_{\geq 1}$ of the denumerable sets $P_n$. It is easy to see that this implies that the set is denumerable (we can display the elements of the union in a double infinite array as in the proof of Proposition 14.2.3).

To deduce that the set of algebraic numbers is denumerable, write the set of all polynomials with integer coefficients as an infinite list $\{p_1, p_2, \ldots\}$ and for $j \in \mathbb{N}$, let $R_j$ be the set whose elements are the roots of the polynomial $p_j$. Then each $R_j$ is finite and the set of algebraic numbers is precisely the
union over all \( j \in \mathbb{N} \) of the sets \( R_j \), which is denumerable.

**Problem 3.**

Prove that if an integer \( n \) is the sum of two perfect squares (\( n = a^2 + b^2 \) for \( a, b \in \mathbb{Z} \)), then \( n \) is of the form \( 4q + r \) for some \( q \in \mathbb{Z} \), where \( r = 0 \), \( r = 1 \) or \( r = 2 \). Deduce that 1234567 cannot be written as the sum of two squares.

**Sol.** Suppose that \( n = a^2 + b^2 \) for some \( a, b \in \mathbb{Z} \). By the division theorem, we have

\[
a = 2q_1 + r_1
\]

and

\[
b = 2q_2 + r_2
\]

for some \( q_1, q_2, r_1, r_2 \in \mathbb{Z} \), with \( r_j \in \{0, 1\}, j = 1, 2 \). Using this, it is easy to find the remainder of \( n = a^2 + b^2 \) after division by 4. The remainders are displayed in the following table.

<table>
<thead>
<tr>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( r(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

We see that the only possible remainders for \( n \) are 0, 1, 2, as required. Finally, since 1234567 = \( 4 \times 308641 + 3 \), it follows from the above that 1234567 is not the sum of two perfect squares.