Problem 1.

Suppose that $X$ and $Y$ are disjoint sets. One can prove that the function

$$\bigcup_{j=0}^{k} \mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y) \rightarrow \mathcal{P}_k(X \cup Y)$$

defined by

$$(A, B) \mapsto A \cup B$$

is a bijection. Taking this for granted, deduce that

$$\left(\begin{array}{c} m + n \\ k \end{array}\right) = \sum_{j=0}^{k} \left(\begin{array}{c} m \\ j \end{array}\right) \left(\begin{array}{c} n \\ k-j \end{array}\right).$$

**Sol.** Let $m, n \in \mathbb{N}$ and let $X$ and $Y$ be disjoint finite sets with $|X| = m, |Y| = n$. Then we have

$$\left|\bigcup_{j=0}^{k} \mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y)\right| = |\mathcal{P}_k(X \cup Y)|$$

since there is a bijection between these two finite sets. By a theorem proved in class, the cardinality in the right-hand side is equal to $\left(\begin{array}{c} m+n \\ k \end{array}\right)$, since

$$|X \cup Y| = |X| + |Y| = m + n$$

(recall that $X$ and $Y$ are disjoint). On the other hand, the sets in the union in the left-hand side are clearly pairwise disjoint so that we have, by the addition and multiplication principles,

$$\left|\bigcup_{j=0}^{k} \mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y)\right| = \sum_{j=0}^{k} |\mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y)|$$

$$= \sum_{j=0}^{k} |\mathcal{P}_j(X)||\mathcal{P}_{k-j}(Y)| = \sum_{j=0}^{k} \left(\begin{array}{c} m \\ j \end{array}\right) \left(\begin{array}{c} n \\ k-j \end{array}\right).$$

Combining this with equation (2) gives the result.

Problem 2.

Prove that for $n \in \mathbb{N}$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0.$$
Sol. We have, by the binomial theorem with \( a = 1 \) and \( b = -1 \),
\[
0 = (1 + (-1))^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j}(-1)^j = \sum_{j=0}^{n} (-1)^j \binom{n}{j}.
\]

Problem 3.

What is the sum of the coefficients of the polynomial \((1 + x)^{53}\)?

Sol. By the binomial theorem, we have
\[
(1 + x)^{53} = \sum_{j=0}^{53} \binom{53}{j} 1^{53-j} x^j = \sum_{j=0}^{53} \binom{53}{j} x^j
\]
hence the sum of the coefficients is
\[
\sum_{j=0}^{53} \binom{53}{j} = 2^{53}.
\]