Problem 1.

Prove by induction on \( n \) that \( n! > 2^n \) for all integers \( n \geq 4 \).

**Sol.** Let us prove by induction on \( n \geq 4 \) the statement \( n! > 2^n \), which we denote by \( P(n) \).

*Base case:* If \( n = 4 \), then \( 4! = 24 \) and \( 2^4 = 16 \). Since \( 24 > 16 \), \( P(4) \) is true.

*Inductive step:* Suppose that \( k! > 2^k \) for some integer \( k \geq 4 \) (inductive hypothesis). We have

\[
(k + 1)! = (k + 1)k! > (k + 1)2^k > (2)2^k = 2^{k+1},
\]

by the induction hypothesis and by the fact that \( k + 1 > 2 \). This shows that \( P(k+1) \) is true, as required.

*Conclusion:* Therefore, by the induction principle, \( n! > 2^n \) for all \( n \geq 4 \).

Problem 2.

Prove by induction on \( n \) that

\[
\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}
\]

for all positive integers \( n \).

**Sol.** We prove by induction on \( n \geq 1 \) the equality, which we denote by \( P(n) \).

*Base case:* For \( n = 1 \), the sum on the left-hand side is equal to the first term, which is \( 1/2 \). The right-hand side is also equal to \( 1/2 \), which shows that \( P(1) \) is true.

*Inductive step:* Assume that

\[
\sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{k}{k+1}
\]

for all \( k \geq 1 \).
for some integer \( k \geq 1 \) (inductive hypothesis). We have
\[
\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \sum_{j=1}^{k} \frac{1}{j(j+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)},
\]
by the induction hypothesis. This last expression is equal to
\[
\frac{k(k+2) + 1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.
\]
Hence \( P(k+1) \) is true.

**Conclusion:** Therefore, by the induction principle, the equality
\[
\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}
\]
holds for all positive integers \( n \).

**Problem 3.**

For a positive integer \( n \), the number \( a_n \) is defined inductively by
\[
a_1 = 1
\]
and
\[
a_{k+1} = \frac{6a_k + 5}{a_k + 2}
\]
for \( k \) a positive integer. Prove by induction on \( n \geq 1 \) that \( 0 < a_n < 5 \).

**Sol.** Let us prove by induction on \( n \geq 1 \) the statement \( 0 < a_n < 5 \), which we denote by \( P(n) \).

**Base case:** For \( n = 1 \), \( a_1 = 1 \) and so clearly \( 0 < a_1 < 5 \). This shows that \( P(1) \) is true.

**Inductive step:** Suppose that \( 0 < a_k < 5 \) for some positive integer \( k \) (inductive hypothesis). First, since \( a_k > 0 \), we have \( 6a_k + 5 > 0 \) and \( a_k + 2 > 0 \), from which it follows that
\[
0 < \frac{6a_k + 5}{a_k + 2} = a_{k+1}.
\]
Secondly, we have
\[
5 > a_{k+1} = \frac{6a_k + 5}{a_k + 2}
\]
if and only if
\[
5a_k + 10 > 6a_k + 5
\]
if and only if \( 5 > a_k \), which is true. This shows that \( 0 < a_{k+1} < 5 \) and thus \( P(k+1) \) is true.
Conclusion: Therefore, by the induction principle, 0 < \(a_n<5\) for all positive integers \(n\).

Problem 4.

Prove by induction on \(n\) that
\[
\prod_{j=2}^{n} \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n}
\]
for integers \(n \geq 2\). See Problem 18 p.55 for the inductive definition of the product.

Sol. Let us prove by induction on \(n \geq 2\) the equality, which we denote by \(P(n)\).

Base case: For \(n = 2\), the product is equal to the first factor, which is \(1 - 1/4 = 3/4\). The right-hand side of the equality is also equal to 3/4, so that \(P(2)\) is true.

Inductive step: Suppose that
\[
\prod_{j=2}^{k} \left(1 - \frac{1}{j^2}\right) = \frac{k+1}{2k}
\]
for some integer \(k \geq 2\) (inductive hypothesis). We have
\[
\prod_{j=2}^{k+1} \left(1 - \frac{1}{j^2}\right) = \prod_{j=2}^{k} \left(1 - \frac{1}{j^2}\right) \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+1}{2k} \left(1 - \frac{1}{(k+1)^2}\right)
\]
\[
= \frac{k+1}{2k} \cdot \frac{(k+1)^2 - 1}{(k+1)^2} = \frac{k^2 + 2k}{2k(k+1)} = \frac{k + 2}{2(k+1)},
\]
which proves \(P(k + 1)\).

Conclusion: Therefore, by the induction principle,
\[
\prod_{j=2}^{n} \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n}
\]
for integers \(n \geq 2\).