Problem 1 (20 pts).

a. Let $A$ and $B$ be two sets. Give the mathematical definitions of $A \cup B$, $A \cap B$ and $A \setminus B$, and illustrate each of these three sets by a Venn diagram.

**Sol.** We have $A \cup B = \{ x : x \in A \text{ or } x \in B \}$, $A \cap B = \{ x : x \in A \text{ and } x \in B \}$ and $A \setminus B = \{ x : x \in A \text{ and } x \notin B \}$. The Venn diagrams can be found in the book, p.69.

b. Let $X$ be a set. Give the definition of $\mathcal{P}(X)$, the power set of $X$.

**Sol.** $\mathcal{P}(X)$ is the set of all subsets of $X$:

$$\mathcal{P}(X) = \{ A : A \subseteq X \}.$$ 

c. If $X = \{1, 2, 3\}$, what is $\mathcal{P}(X)$?

**Sol.** We have

$$\mathcal{P}(X) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \}.$$ 

d. If $A$ and $B$ are two subsets of some universal set $U$, write $(A \cup B)^c$ and $(A \cap B)^c$ in terms of $A^c$ and $B^c$. In other words, state the De Morgan laws. No justification is needed here.

**Sol.** We have $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. 

Problem 2 (20 pts).

a. Give the definition of the binomial coefficient $\binom{n}{r}$, where $r$ and $n$ are two non-negative integers.

Sol. The binomial coefficient $\binom{n}{r}$ is the cardinality of $\mathcal{P}_r(X)$ when $|X| = n$, i.e. it is the number of subsets of a set of cardinality $n$ which have cardinality $r$.

b. What is the relationship between the two binomial coefficients $\binom{132}{37}$ and $\binom{132}{95}$? Give a short justification.

Sol. They are equal, since
\[
\binom{n}{r} = \binom{n}{n-r}
\]
for any integers $n, r$ with $0 \leq r \leq n$.

c. State the Binomial Theorem.

Sol. The theorem states that for all real numbers $a$ and $b$ and non-negative integers $n$, we have
\[
(a + b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j.
\]
Problem 3 (20 pts).

a. Let \( a \) and \( b \) be positive integers, and suppose that there exist integers \( m \) and \( n \) such that 
\[
ma + nb = 1.
\]
Prove that \( (a, b) = 1 \).

Solutions.
If \( d = (a, b) \), then \( d \) divides both \( a \) and \( b \), so that it must divide \( ma + nb = 1 \). It follows that \( d = 1 \).

b. Let \( a \) and \( b \) be positive integers, and let \( m \) and \( n \) be integers such that 
\[
ma + nb = d,
\]
where \( d = (a, b) \). Prove that \( (m, n) = 1 \).

Solutions.
From \( ma + nb = d \) we get 
\[
\frac{ma}{d} + \frac{nb}{d} = 1.
\]
Since \( a/d \) and \( b/d \) are integers (\( d \) is a common divisor of \( a \) and \( b \)), it follows from part a. that \( (m, n) = 1 \).

c. Find all solutions \( m, n \) to the diophantine equation 
\[
133m + 56n = 35.
\]

Solutions.
With the euclidean algorithm, we find that \( (133, 56) = 7 \). Since 7 divides 35, the equation has solutions. Proceeding backwards in the euclidean algorithm, we find that 
\[
3 \times 133 - 7 \times 56 = 7,
\]
so that \( m_0 = 15 \) and \( n_0 = -35 \) is a solution to 
\[
133m_0 + 56n_0 = 35.
\]
It follows from a theorem proved in class that all solutions \( m \) and \( n \) are given by 
\[
m = m_0 + \frac{b}{(a, b)} q = 15 + 8q
\]
and 
\[
n = n_0 - \frac{a}{(a, b)} q = -35 - 19q
\]
for some \( q \in \mathbb{Z} \).
Problem 4 (20 pts).

a. Prove that 7 divides \(3 \cdot 2^{101} + 9\).

**Sol.** Since \(2^3 \equiv 1 \mod 7\), we have that
\[
3 \cdot 2^{101} + 9 = 3 \cdot (2^3)^{33} \cdot 2^2 + 9 \equiv 3 \cdot 2^2 + 9 \mod 7 \equiv 21 \mod 7 \equiv 0 \mod 7,
\]
which proves the result.

b. Prove that a positive integer \(n\) is divisible by 9 if and only if the sum of its digits is divisible by 9.

*(Hint: Write \(n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10 a_1 + a_0\), where \(a_0, a_1, \ldots, a_{k-1}, a_k\) are the digits of \(n\).)*

**Sol.** Since \(10 \equiv 1 \mod 9\), we have
\[
n \equiv 1^k a_k + 1^{k-1} a_{k-1} + \cdots + 1 a_1 + a_0 \mod 9 \equiv a_k + a_{k-1} + \cdots + a_1 + a_0 \mod 9,
\]
so that \(n\) is divisible by 9 if and only if \(a_k + a_{k-1} + \cdots + a_1 + a_0\) is divisible by 9, as required.
Problem 5 (20 pts).

a. Define what is a prime number, and state the Fundamental Theorem of Arithmetic.

**Sol.** A positive integer $n$ is prime if $n > 1$ and the only positive divisors of $n$ are 1 and $n$. The Fundamental Theorem of Arithmetic states that every positive integer greater than 1 can be written uniquely as a product of prime numbers, with the prime factors in the product written in non-decreasing order.

b. Prove by contradiction that there are infinitely many prime numbers. 
*(Hint: Suppose that $p_1, p_2, \ldots, p_n$ are the only prime numbers, and consider the number $m = p_1 p_2 \cdots p_n + 1$.)

**Sol.** See Theorem 23.5.1, p.285