Solution: MAT 131 Summer 2015 Midterm I

1Write down functions with respective requirements:

a) Write down a function with domain $(-\infty, 0) \cup (0, +\infty)$ and range $(2, 3)$.(Indicate your domain) (5 pts)

solution:

$$f(x) = \begin{cases} 2.5 & x \geq 1 \\ x+2 & x \in (0,1) \\ 2.5 & x < 0 \end{cases}$$

b) Write down a function that is one-to-one with domain $(-2, +\infty)$.(Indicate your domain)(5 points)

solution:

$$f(x) = x, x \in (-2, +\infty)$$

c) Write down a function with domain $(-\infty, +\infty)$ that satisfies the following two conditions:(5 points)

i) $\lim_{x \to 0} f(x)$does not exist;

ii) $\lim_{x \to 1} f(x) = 1$;

solution:

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

d) Write down a function with domain $(-\infty, +\infty)$ that satisfies the following two conditions:(5 points)

i) $\lim_{x \to 0^+} f(x) = 1$;

ii) $\lim_{x \to 0^-} f(x) = 0$;

solution:

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

2 Find the domain of the following functions:

a) $f(x) = \ln(-x^2 + 4x - 5)$;(10 points)

solution:

The domain for the function $\ln x$ is $(0, +\infty)$, so we get : $-x^2+4x-5 > 0$, solution for this inequality is empty,i.e, the domain is $\emptyset$.

b) $f(x) = \frac{2x}{x^4-3x^2+2}$;(10 points)

solution:

The denominator of this function should not be zero, thus we solve for: $x^4 - 3x^2 + 2 = 0$; to solve this ,set $t = x^2$,we will have $t^2 - 3t + 2 = 0$; and we get $t = 1$ or $t = 2$; thus $x = \pm 1$,$\pm\sqrt{2}$; so we have the domain $(-\infty, -\sqrt{2}) \cup (-\sqrt{2}, -1) \cup (-1, 1) \cup (1, \sqrt{2}) \cup (\sqrt{2}, \infty)$.

3 Sketch the graph of the function $f(x) = |-x^2 + 3x - 2|$(indicate the x-intercept and y-intercept);(10 points)

solution:

The y-intercept: $y = f(0) = 2$ The x-intercept: From $f(x) = 0$ we can get

$$-x^2 + 3x - 2 = 0$$

;the solution is $x = 1, 2$;

Draw the picture for $y = -x^2 + 3x - 2$, then flap the part below x-axis .

4 Find the limit (10 points)

$$\lim_{x\to 2} \frac{x^3 + 4x - 16}{x - 2}$$

solution: set $x = t + 2$,then

$$\frac{x^3 + 4x - 16}{x - 2} = \frac{t^3 + 6t^2 + 16t}{t} = t^2 + 6t + 16$$

so we get

$$\lim_{t\to 0} (t^2 + 6t + 16) = 16$$

5 If $x, y \in (0, \pi/2)$, $\sin x = 1/3, \cos y = 1/2$,compute $\cos(x + y)$.(10 points) (The following formulas might be useful: $\sin(x+y) = \sin x \cos y + \sin y \cos x$, $\cos(x + y) = \cos x \cos y - \sin x \sin y$)

solution:

$$(\sin x)^2 + (\cos x)^2 = 1$$

so we have $\cos x = \frac{2\sqrt{2}}{3}$ since $x \in (0, \pi/2)$; likewise we have $\sin y = \frac{\sqrt{3}}{2}$; so we have

$$\cos(x+y) = \frac{2\sqrt{2}}{3} \times \frac{1}{2} - \frac{1}{3} \times \frac{\sqrt{3}}{2} = \frac{2\sqrt{2} - \sqrt{3}}{6}$$

6 Find the inverse function for $f(x) = 2^{\sin x + 1}, x \in (0, \pi/2)$,and what's the range for $f^{-1}(x)$?(10 points)

solution:

Since $y = 2^{\sin x + 1}$, then we have $\log_2 y = \sin x + 1$; so $\sin x = \log_2 y - 1$; therefore we have $x = \arcsin(\log_2 y - 1)$; so

$$f^{-1}(x) = \arcsin(\log_2 x - 1)$$

The range of $f^{-1}(x)$ is the domain of $f(x)$, that is, $(0, \pi/2)$.

7 $f(x) = C \cdot e^{kx} + \ln(1+x)$, where C, k are constants;and $f(0) = 1, f(1) = 1 + \ln 2$;

a) find the value of C, k;(10 points)

b) find the domain for $f(x)$;(5 points)

c) Suppose we already know that the inverse function exists, find $f^{-1}(x)$.(5 points)

solution:

a)

$$1 = f(0) = C \cdot e^0 + \ln 1 = C + 0 = C$$

Then we get

$$1 + \ln 2 = f(1) = 1 \cdot e^k + ln2 = e^k + \ln 2$$

that is, $1 = e^k$,so $k = 0$.

b)Since $f(x) = 1 \cdot e^0 + \ln(1+x) = 1 + \ln(1+x)$, we have the domain $x \in (-1, +\infty)$.

c)Since $y = 1 + \ln(1+x)$,then $y - 1 = \ln(1+x)$, so $x = e^{y-1} - 1$, i.e, $f^{-1}(x) = e^{x-1} - 1$.

bonus Suppose the domain for $f(x)$ is $(-1, 1)$, and the range is $(2, \pi)$,and

$$\lim_{x \to 0} f(x) = 2$$

find the limit (10 points)

$$\lim_{x\to 0}(\frac{1}{f(x)-2}+\frac{2}{f(x)^2-6f(x)+8})$$

solution:

First we simplify the function when $t \neq 2$,
$\frac{1}{t-2}+\frac{2}{t^2-6t+8}=\frac{1}{t-2}+\frac{2}{(t-2)(t-4)}=\frac{(t-4)+2}{(t-2)(t-4)}=\frac{1}{t-4}$

and Since $f(x) \in (2,\pi)$, thus $f(x)-2 \neq 0$, thus we can simplify the function to be $\frac{1}{f(x)-4}$

So we have

$$\lim_{x\to 0}\frac{1}{f(x)-4}=-1/2$$