Solution: MAT 131 Summer 2015 Midterm I

1 Write down functions with respective requirements:

a) Write down a function with domain \((-\infty, 0) \cup (0, +\infty)\) and range \((2, 3)\). (Indicate your domain) (5 pts)

solution:

\[ f(x) = \begin{cases} 
2.5 & x \geq 1 \\
 x + 2 & x \in (0, 1) \\
2.5 & x < 0 
\end{cases} \]

b) Write down a function that is one-to-one with domain \((-2, +\infty)\). (Indicate your domain) (5 pts)

solution:

\[ f(x) = x, x \in (-2, +\infty) \]

c) Write down a function with domain \((-\infty, +\infty)\) that satisfies the following two conditions: (5 points)

i) \(\lim_{x \to 0} f(x)\) does not exist; 

ii) \(\lim_{x \to 1} f(x) = 1;\)

solution:

\[ f(x) = \begin{cases} 
0 & x < 0 \\
1 & x \geq 0 
\end{cases} \]

d) Write down a function with domain \((-\infty, +\infty)\) that satisfies the following two conditions: (5 points)

i) \(\lim_{x \to 0^+} f(x) = 1;\)

ii) \(\lim_{x \to 0^-} f(x) = 0;\)

solution:

\[ f(x) = \begin{cases} 
0 & x < 0 \\
1 & x \geq 0 
\end{cases} \]

2 Find the domain of the following functions:

a) \(f(x) = \ln(-x^2 + 4x - 5);\) (10 points)
The domain for the function \( \ln x \) is \((0, +\infty)\), so we get: \(-x^2 + 4x - 5 > 0\), solution for this inequality is empty, i.e., the domain is \( \emptyset \).

b) \( f(x) = \frac{2x}{x^2 - 3x^2 + 2} \); (10 points)

solution:

The denominator of this function should not be zero, thus we solve for:

\( x^4 - 3x^2 + 2 = 0 \); to solve this, set \( t = x^2 \), we will have \( t^2 - 3t + 2 = 0 \); and we get \( t = 1 \) or \( t = 2 \); thus \( x = \pm 1, \pm \sqrt{2} \); so we have the domain \((-\infty, -\sqrt{2}) \cup (-\sqrt{2}, -1) \cup (-1, 1) \cup (1, \sqrt{2}) \cup (\sqrt{2}, \infty)\).

3 Sketch the graph of the function \( f(x) = | -x^2 + 3x - 2| \) (indicate the x-intercept and y-intercept); (10 points)

solution:

The y-intercept: \( y = f(0) = 2 \) The x-intercept: From \( f(x) = 0 \) we can get

\[-x^2 + 3x - 2 = 0\]; the solution is \( x = 1, 2 \);

Draw the picture for \( y = -x^2 + 3x - 2 \), then flap the part below x-axis.

4 Find the limit (10 points)

\[
\lim_{x \to 2} \frac{x^3 + 4x - 16}{x - 2}
\]

solution: set \( x = t + 2 \), then

\[
\frac{x^3 + 4x - 16}{x - 2} = \frac{t^3 + 6t^2 + 16t}{t} = t^2 + 6t + 16
\]

so we get

\[
\lim_{t \to 0} (t^2 + 6t + 16) = 16
\]

5 If \( x, y \in (0, \pi/2) \), \( \sin x = 1/3 \), \( \cos y = 1/2 \), compute \( \cos(x + y) \). (10 points)

(The following formulas might be useful: \( \sin(x + y) = \sin x \cos y + \sin y \cos x \), \( \cos(x + y) = \cos x \cos y - \sin x \sin y \))

solution:

\[
(\sin x)^2 + (\cos x)^2 = 1
\]
so we have \( \cos x = \frac{2\sqrt{2}}{3} \) since \( x \in (0, \pi/2) \); likewise we have \( \sin y = \frac{\sqrt{3}}{2} \); so we have

\[ \cos(x + y) = \frac{2\sqrt{2}}{3} \times \frac{1}{2} - \frac{1}{3} \times \frac{\sqrt{3}}{2} = \frac{2\sqrt{2} - \sqrt{3}}{6} \]

6 Find the inverse function for \( f(x) = 2^{\sin x+1}, x \in (0, \pi/2) \), and what’s the range for \( f^{-1}(x) \)? (10 points)

**solution:**

Since \( y = 2^{\sin x+1} \), then we have \( \log_2 y = \sin x + 1 \); so \( \sin x = \log_2 y - 1 \); therefore we have \( x = \arcsin(\log_2 y - 1) \); so

\[ f^{-1}(x) = \arcsin(\log_2 x - 1) \]

The range of \( f^{-1}(x) \) is the domain of \( f(x) \), that is, \((0, \pi/2)\).

7 \( f(x) = C \cdot e^{kx} + \ln(1 + x) \), where \( C, k \) are constants; and \( f(0) = 1, f(1) = 1 + \ln 2 \);

a) find the value of \( C, k \); (10 points)
b) find the domain for \( f(x) \); (5 points)
c) Suppose we already know that the inverse function exists, find \( f^{-1}(x) \). (5 points)

**solution:**

a)

\[ 1 = f(0) = C \cdot e^0 + \ln 1 = C + 0 = C \]

Then we get

\[ 1 + \ln 2 = f(1) = 1 \cdot e^k + \ln 2 = e^k + \ln 2 \]

that is, \( 1 = e^k \), so \( k = 0 \).

b) Since \( f(x) = 1 \cdot e^0 + \ln(1 + x) = 1 + \ln(1 + x) \), we have the domain \( x \in (-1, +\infty) \).

c) Since \( y = 1 + \ln(1 + x) \), then \( y - 1 = \ln(1 + x) \), so \( x = e^{y-1} - 1 \), i.e., \( f^{-1}(x) = e^{x-1} - 1 \).

**bonus** Suppose the domain for \( f(x) \) is \((-1, 1)\), and the range is \((2, \pi)\), and

\[ \lim_{x \to 0} f(x) = 2 \]
find the limit (10 points)

\[
\lim_{x \to 0} \left( \frac{1}{f(x) - 2} + \frac{2}{f(x)^2 - 6f(x) + 8} \right)
\]

solution:

First we simplify the function when \( t \neq 2 \),

\[
\frac{1}{t-2} + \frac{2}{t^2-6t+8} = \frac{1}{t-2} + \frac{2}{(t-2)(t-4)} = \frac{(t-4)+2}{(t-2)(t-4)} = \frac{1}{t-4}
\]

and Since \( f(x) \in (2, \pi) \), thus \( f(x) - 2 \neq 0 \), thus we can simplify the function to be \( \frac{1}{f(x) - 4} \)

So we have

\[
\lim_{x \to 0} \frac{1}{f(x) - 4} = -1/2
\]