Math 131
Summer 2015
Exam 3
6/25/15
Time Limit: 120 Minutes

Name (Print): ____________________

ID number ____________________

This exam contains 10 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	30	
2	20	
3	10	
4	30	
5	10	
6	10	
Total:	110	

1. (Find the limit)

   (a) (5 points)

$$\lim_{x\to 0}\frac{1-\cos 2x}{1-\cos x}$$

$$\lim_{x\to 0}\frac{1-\cos 2x}{1-\cos x}=\lim_{x\to 0}\frac{2\sin 2x}{\sin x}=\lim_{x\to 0}\frac{4\cos 2x}{\cos x}=4$$

   (b) (5 points)

$$\lim_{x\to +\infty}(1+\frac{2}{x})^x$$

$$\lim_{x\to +\infty}(1+\frac{2}{x})^x=\lim_{x\to +\infty}e^{x\ln(1+\frac{2}{x})}$$

And Since

$$\lim_{x\to +\infty}x\ln(1+\frac{2}{x})=\lim_{x\to +\infty}\frac{\ln(1+\frac{2}{x})}{\frac{1}{x}}=\lim_{\to +\infty}\frac{\frac{1}{1+\frac{2}{x}}\cdot\frac{-2}{x^2}}{\frac{-1}{x^2}}=2$$

So

$$\lim_{x\to +\infty}(1+\frac{2}{x})^x=e^2$$

(c) (5 points)

$$\lim_{x\to 0}\frac{\sin 2x}{\tan 3x}$$

$$\lim_{x\to 0}\frac{\sin 2x}{\tan 3x}=\lim_{x\to 0}\frac{2\cos 2x}{3(\sec 3x)^2}=\frac{2}{3}$$

(d) (5 points)

$$\lim_{x\to 1}\frac{x^9-1}{x^4-1}$$

$$\lim_{x\to 1}\frac{x^9-1}{x^4-1}=\lim_{x\to 1}\frac{9x^8}{4x^3}=\frac{9}{4}$$

(e) (5 points)

$$\lim_{x\to 0} x \cdot \cot 2x$$

$$\lim_{x\to 0} x \cdot \cot 2x = \lim_{x\to 0} \frac{x}{\tan 2x} = \lim_{x\to 0} \frac{1}{2(\sec 2x)^2} = \frac{1}{2}$$

(f) (5 points)

$$\lim_{x\to 0^+} x^{\sin x}$$

$$\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} e^{\sin x \ln x}$$

And Since

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\cos x}{(\sin x)^2}} = \lim_{x\to 0^+} \frac{-(\sin x)^2}{x\cos x} = \lim_{x\to 0^+} \frac{-2\sin x\cos x}{\cos x - x\sin x} = 0$$

So

$$\lim_{x\to 0^+} x^{\sin x} = e^0 = 1$$

2. If $f(x) = x^3 - 2x^2 + x + 1$.

   (a) (5 points) On what intervals are f increasing and on what intervals are f decreasing?

   (b) (5 points) On what intervals are f concave upward and on what intervals are f concave downward?

   (c) (5 points) What's the local extreme values?

   (d) (5 points) Sketch the graph of the function f.(Indicate monotonicity, concavity,local extreme)

   This question is exactly the same question from last exam,check answers there

3. Find the derivatives

(a) (5 points)

$$f(x) = \ln(1 - \ln(1 - \ln x))$$

$$f'(x) = \frac{1}{1 - \ln(1 - \ln x)} \cdot \frac{-1}{1 - \ln x} \cdot \frac{-1}{x}$$

(b) (5 points)

$$f(x) = \sin(x^x)$$

$$f'(x) = \cos(x^x) \cdot x^x \cdot (1 + \ln x)$$

4. Find the the global extreme values on the given interval(may not exist):

   (a) (10 points)

$$f(x) = e^{\sin x}, x \in [0, 2\pi]$$

$$f^{'}(x) = e^{\sin x} \cdot \cos x, x \in [0, 2\pi]$$

Set

$$f^{'}(x) = 0$$

$\Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

Since

$$f(0) = f(2\pi) = e^0 = 1$$

$$f(\frac{\pi}{2}) = e^1 = e$$

$$f(\frac{3\pi}{2}) = \frac{1}{e}$$

So the global Max Value is e, and global Min Value is $\frac{1}{e}$.

Remark: Here we do not have to determine weather the function at a given critical point is max or min or neither. We only need to compare them since we just need to know the global extreme values.

   (b) (10 points)

$$f(x) = \frac{\ln x}{x^2}, x \in [\frac{1}{e}, 1]$$

$$f^{'}(x) = \frac{1 - 2\ln x}{x^3} > 0$$

Since $x \in [\frac{1}{e}, 1]$ So f is increasing on the domain. Thus we have Max=$f(1) = 0$, and Min=$f(\frac{1}{e}) = -e^2$.

(c) (10 points)

$$f(x) = x + \frac{1}{x}, x \in (0, +\infty)$$

$$f^{'}(x) = 1 - \frac{1}{x^2}, f^{''}(x) = \frac{2}{x^3}$$

So set $f^{'}(x) = 0 \Rightarrow x = 1$,and Since $f^{''}(1) > 0$. So $f(1) = 2$ is local min. And

$$\lim_{x \to 0} x + \frac{1}{x} = +\infty$$

$$\lim_{x \to +\infty} x + \frac{1}{x} = +\infty$$

This means:

i)There is no global max.

ii) Global min exists, and it is one of the local minimums. So that is $f(1) = 2$.

5. Find $f(x)$.

(a) (5 points)

$$f^{'}(t) = 2t - \sin t, f(0) = 3$$

(b) (5 points)

$$f^{''}(t) = 1 - 2t, f(0) = 0, f(1) = \frac{7}{6}$$

6. **Bonus**

(a) (10 points) A cone-shaped drinking cup made from a circular piece of paper of radius R by cutting out a sector and joining the edges. Find the maximum capacity of such a cup. (The formula for the volume of cone:$V = \frac{1}{3}S \cdot h$)

Suppose the height of the cone is h and the radius of the base circle is r. We have the following relation:

$$r^2 + h^2 = R^2$$

So

$$V(h) = \frac{1}{3} \cdot \pi r^2 h = \frac{\pi h(R^2 - h^2)}{3}, h \in [0, R]$$

$$V^{'}(h) = \pi(\frac{R^2}{3} - h^2)$$

Set $V^{'}(h) = 0 \Rightarrow h = \frac{R}{\sqrt{3}}$. And

$$V(0) = V(R) = 0$$

$$V\left(\frac{R}{\sqrt{3}}\right) = \frac{2\sqrt{3}\pi R^3}{27}$$

So the max capacity is $\frac{2\sqrt{3}\pi R^3}{27}$.