MAT322 Spring 2016 Midterm 2

Name: ________________________________ SB ID number: ______________________

Problem 1: ______ /25 **Problem 2**: ______ /25 **Problem 3**: ______ /25

Problem 4: ______ /25 **Problem 5**: ______ /25 **Problem 6**: ______ /25

Total: ______ /100

Instructions: The exam is closed book, closed notes, calculators are not allowed, and all cellphones and other electronic devices must be turned off for the duration of the exam. You will have approximately 90 minutes for this exam. You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., then **please raise your hand**.
You must attempt Problem 1. Of the remaining five problems, you should attempt **three**. If you choose to do more than three of the remaining five problems, you will get the highest three scores. You may quote results from the book, but you must give a clear reference, e.g., "the criterion for Riemann integrability" or "the C^1 criterion for differentiability".

Name: ________________________ **Problem 1**: ____________ /25

Problem 1. Mandatory Problem.(25 points) Let a be a positive real number. Let U be the open subset $(-\pi, \pi) \times (-a, a) \subset \mathbb{R}^2$. Define $\alpha : U \to \mathbb{R}^3$ by $\alpha(\theta, r) = (r\cos(\theta), r\sin(\theta), \theta)$. Please show all work.

(a)(10 points) With respect to the standard ordered basis $(\mathbf{e}_1, \mathbf{e}_2)$ for $\mathbb{R}^2$ and with respect to the standard ordered basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for $\mathbb{R}^3$, compute the matrix representative A of the total derivative, $D_{(r,\theta)}\alpha$.

(b)(8 points) Consider the transpose $A^\dagger$. Showing all work, compute the column space of $A^\dagger$ and compute the kernel of $A^\dagger$. In particular, state the dimension of the column space, and state the dimension of the kernel.

(c)(7 points) Compute the matrix $A^\dagger A$, and compute the determinant of $A^\dagger A$.

Name: ______________________________ **Problem 2**: ____________ /25

Problem 2(25 points) This problem continues the previous problem. Please show all work.

(a)(8 points) Check that α is one-to-one.

(b)(7 points) Compute $|\mathrm{vol}_{\mathbb{R}^3,2}(D_{(\theta,r)}\alpha)|$.

(c)(10 points) Set up the integral to compute $\mathrm{vol}_{\mathbb{R}^3,2}(\alpha(U))$. Then use Fubini's Theorem to reduce your integral to a single variable integral $\int g(r)dr$. You need not evaluate this last integral.

Name: ________________________________ **Problem 3**: ______________ /25

Problem 3(25 points) First, state carefully a condition on a subset M of $\mathbb{R}^m$ that it is an embedded k-dimensional manifold with boundary of class C^r. Then check that the following subset of $\mathbb{R}^4$ is an embedded 3-dimensional manifold with boundary. Explain your answer.

$$M = \{(s,t,u,v) \in \mathbb{R}^4 : st + uv + 1 = 0,\ s^3 + t^3 + u^3 + v^3 \geq 0\}.$$

Name: ______________________________ **Problem 4**: ____________ /25

Problem 4(25 points) Perform each of the following computations with tensors on $V = \mathbb{R}^4$.

(a)(7 points) For the 2-tensors $f = \phi_{1,2} + \phi_{3,4}$ and $g = \phi_{1,3} - \phi_{2,4}$, compute $h = f \otimes g$ as a linear combination of standard basis 4-tensors, and then write out $h(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$ as a function of the coordinates.

(b)(8 points) For the tensor h from (a), compute the following alternating 4-tensor,

$$\omega(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \sum_{\sigma \in S_4} \operatorname{sign}(\sigma) h(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}).$$

Express your answer as a linear combination of the standard ordered basis for $\mathcal{A}^4(V)$, ψ_{i_1,i_2,i_3,i_4}, $i_1 < i_2 < i_3 < i_4$, where

$$\psi_{i_1,i_2,i_3,i_4} = \sum_{\sigma \in S_4} \operatorname{sign}(\sigma) \phi_{i_{\sigma(1)}} \otimes \phi_{i_{\sigma(2)}} \otimes \phi_{i_{\sigma(3)}} \otimes \phi_{i_{\sigma(4)}}.$$

(b)(10 points) For the standard ordered basis of $\mathcal{A}^2(V)$, $\mathcal{B} = (\psi_{1,2}, \psi_{1,3}, \psi_{1,4}, \psi_{2,3}, \psi_{2,4}, \psi_{3,4})$, $\psi_{i,j} = \phi_i \wedge \phi_j$, compute the matrix representative $[T^*]_{\mathcal{B},\mathcal{B}}$ of the $\mathbb{R}$-linear transformation $T^* : \mathcal{A}^2(V) \to \mathcal{A}^2(V)$ corresponding to the linear transformation $T(\mathbf{e}_i) = \mathbf{e}_{i+1}$, $i = 1, 2, 3$, and $T(\mathbf{e}_4) = \mathbf{e}_1$. Also write out $T^*(\psi_{1,3})$ as a linear combination of the elements of $\mathcal{B}$.

Name: ______________________________ **Problem 5**: ____________ /25

Problem 5(25 points) Let a and b be positive real numbers. Let U be the open subset $\{(s,t) \in \mathbb{R}^2 : s^2 + t^2 < a^2\} \subset \mathbb{R}^2$. Define $\alpha : U \to \mathbb{R}^4$ by $\alpha(s,t) = (s, t, (s^2 - t^2)/b, 2st/b)$. Please show all work.

(a)(5 points) With respect to the standard ordered basis $(\mathbf{e}_1, \mathbf{e}_2)$ for $\mathbb{R}^2$ and with respect to the standard ordered basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ for $\mathbb{R}^4$, compute the matrix representative A of the total derivative $D_{(s,t)}\alpha$.

(b)(5 points) Explain why α is a diffeomorphism to its image.

(c)(7 points) Compute $|\mathrm{vol}_{\mathbb{R}^4,2}(D_{(s,t)}\alpha)|$. Simplify your answer as much as possible, expressed in terms of elementary functions of s and t.

(d)(8 points) Compute the volume of $\alpha(U)$. Please evaluate your integral. Express your final answer in terms of elementary functions of a and b.

Name: ________________________________ **Problem 6**: ______________ /25

Problem 6(25 points) Let m and k be integers with $1 \leq k \leq m-1$. Denote by $C_{m,k}$ the set of $\binom{m}{k}$ ordered k-tuples $I = (i_1, \dots, i_k)$ of integers with $1 \leq i_1 < i_2 < \cdots < i_k \leq m$. For every $I \in C_{m,k}$, denote by $\pi_I : \mathbb{R}^m \to \mathbb{R}^k$ the linear projection $(x_1, \dots, x_m) \mapsto (x_{i_1}, \dots, x_{i_k})$.

Let $M \subset \mathbb{R}^m$ be an embedded, k-dimensional submanifold of class C^r, $r \geq 1$, such that M is compact and with empty boundary. Prove that there exists a collection of pairs $(U_I, \alpha_I)_{I \in C_{m,k}}$ of open subsets, $U_I \subset \mathbb{R}^k$, and diffeomorphisms of class C^r, $\alpha_I : U_I \to \alpha_I(U_I)$, with $(\alpha_I(U_I))_{I \in C_{m,k}}$ a covering of M by (relatively) open subsets and such that on each connected component $U_{I,j}$ of U_I, $\pi_I \circ \alpha_I : U_{I,j} \to \mathbb{R}^k$ equals translation by a vector in $\mathbb{R}^k$.