MAT 200 Fall 2019
Midterm 2

November 25 2019

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Instructions

- This examination contains 6 problems.
- You have **80 minutes** to complete the examination.
- Write your answers neatly. Use the empty space on the back of the questions for rough working.
- You may use one double-sided 8.5” $\times$ 11” pages with notes that you have prepared. You may not use any other resources, including lecture notes, books, other students or other engineers.
- You may not use a calculator.

Problem 1 (10 points)

(a) Use Euclid's algorithm to find the greatest common divisor of 3333 and 1122.

(b) Can you find two integers $m, n \in \mathbb{Z}$ such that

$$3333m + 1122n = 33?$$

If so, find them (and show your work), if not, explain why not.

(c) Can you find two integers $m, n \in \mathbb{Z}$ such that

$$3333m + 1122n = 3?$$

If so, find them (and show your work), if not, explain why not.

(d) Can you find two integers $m, n \in \mathbb{Z}$ such that

$$3333m + 1122n = 99?$$

If so, find them (and show your work), if not, explain why not.

Problem 2 (10 points)

Let X, Y, Z be sets (possibly infinite). We recently defined what that the symbols $\{\leq, <, =\}$ mean in the context of comparing 'sizes' of (possibly infinite) sets. In this problem we will prove that many of the familiar properties from the finite setting, still hold in the infinite setting.

(a) Suppose $|X| = |Y|$ and $|Y| = |Z|$. Prove that $|X| = |Z|$.

(b) Suppose $|X| = |Y|$. Prove that $|X| \leq |Y|$.

(c) Suppose $|X| \leq |Y|$ and $|Y| \leq |Z|$. Prove that $|X| \leq |Z|$.

Problem 3 (5 points)

The following boxed text is an attempt to show that the square root of 4 is irrational. There must be at least one mistake somewhere because we already know that the square root of 4 is rational.

Circle the first logically incorrect step and explain why it is incorrect.

Suppose for contradiction that p, q are integers such that

$$\left(\frac{p}{q}\right)^2 = 4.$$

Let $\tilde{p}, \tilde{q}$ be the reduced numerator and denominator corresponding to $\frac{p}{q}$, meaning we have cancelled out all the common factors from p and q so that we still have

$$\left(\frac{\tilde{p}}{\tilde{q}}\right)^2 = 4,$$

but $\tilde{p}$ and $\tilde{q}$ have no factors in common. We can rearrange this to get

$$\tilde{p}^2 = 4\tilde{q}^2. \tag{1}$$

It follows that $\tilde{p}^2$ is divisible by 4, since the right hand side is divisible by 4. It follows that $\tilde{p}$ is also divisible by 4.
Therefore $\tilde{p} = 4r$ for some integer r, and we can substitute this into (1) to get

$$16r^2 = 4\tilde{q}^2,$$

which implies (dividing both sides by 4)

$$4r^2 = \tilde{q}^2.$$

Therefore, by the same logic as before, $\tilde{q}^2$ is divisible by 4, and $\tilde{q}$ is also divisible by 4.
This is a contradiction, because we already cancelled out all the common factors between $\tilde{p}$ and $\tilde{q}$, yet we have just shown that both of them are divisible by 4.

Problem 4 (10 points)

Recall that $\mathrm{Fun}(X, Y)$ is the set of all functions from X to Y, and $\mathbb{N}_n = \{1, 2, \ldots, n\}$, and $P(X)$ is the set of all subsets of X.

(a) Suppose Y has m elements. How many elements does the set $\mathrm{Fun}(\mathbb{N}_3, Y)$ have? (No justification needed, just write your answer).

(b) Give an example of a function with domain $\mathrm{Fun}(\mathbb{N}_3, Y)$ and codomain $P(Y)$.

(c) Suppose Y has m elements. How many elements does the set $\mathrm{Fun}(\ \mathrm{Fun}(\mathbb{N}_3, Y), P(Y)\ )$ have? (No justification needed, just write your answer).

(d) Suppose Y has m elements and $m \geq 2$. How many injective functions are there from $\mathrm{Fun}(\mathbb{N}_3, Y)$ to $P(Y)$? Justify your answer.

(e) Consider the following set.

$$\mathrm{Fun}(X, Y) \cup (\mathbb{N}_3 \times Y) \cup (\{3, 4, 5\} \times Y)$$

How many elements are in this set? Your answer should be in terms of m and n. (No justification needed).

Problem 5 (10 points)

Let X be a set. Find a bijection between the sets 2^X and $\mathrm{Fun}(X, \{2, 6\})$. Prove that it is a bijection.

Problem 6 (15 points)

(a) What is the remainder of $11^{999,999,999,999,999}$ when divided by 3?

(b) In class we proved that

$$n \text{ is a square} \implies \text{The remainder of } n \text{ when divided by 3 is: 0 or 1.} \qquad (2)$$

State and prove an analogous theorem for division by 11 instead of 3. *Hint: Use modular arithmetic.*

(c) Is $11^{999,999,999,999,999} + 2$ a perfect square? If so, write down the square root. If not, explain why not.

(d) Is $11^{999,999,999,999,999} + 3$ a perfect square? If so, write down the square root. If not, explain why not.