Curves and surfaces

Moira Chas

Question 1 *How would you define a* surface*? Can you give examples of surfaces?*

Answers:

Remark You probably know what the word *surface* means in everyday English. Mathematicians use the word surface with a different (but related) meaning.

Examples of surfaces (a) A very thin sheet of paper.

(b) A very thin sheet of paper glued according the recipe below.

(c) The skin of a solid object (but not the solid object itself!).

A recipe to make surfaces (1) Cut polygons out of paper.

(2) Choose pairs of straight edges of the same length and *sew* them together (with tape.)

(3) The pieces of paper should not overlap (see Figure 1).

(4) Once a pair of edges has been sewed, it becomes "just one edge" and CANNOT be used again. Hence, at the end of the process, any edge can belong to one or two polygons, but not to three or more.

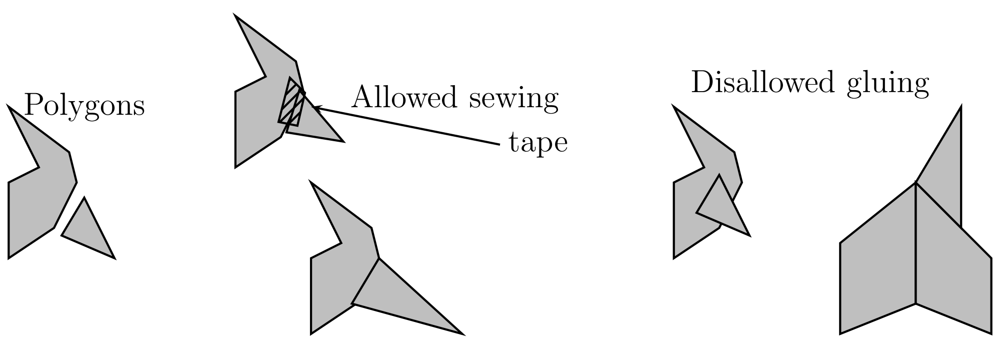


Figure 1: Recipe for constructing surfaces

Question 2 *Which of the rules of the recipe for making surfaces are broken by the disallowed gluing of Figure 1?*

Experiment 1 Using the above recipe, construct two surfaces:

(1) one made of at least four polygons and gluing at least seven pairs of edges.
(2) one using only one polygon and gluing at least four pairs of edges.

□

Experiment 2 Using the above recipe, construct two surfaces:

(1) one which is not the skin of any solid.
(2) one which is the skin of a solid.

What can you say about these two surfaces? □

Exercise 1 *Determine whether each of the objects of Figure 2 is a surface.*

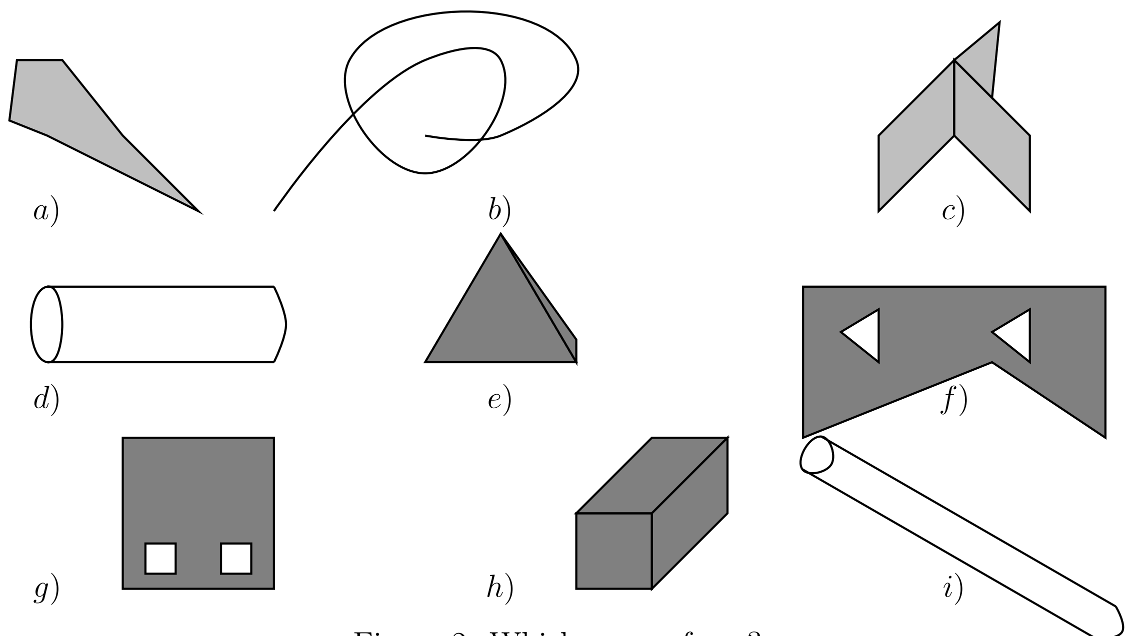


Figure 2: Which are surfaces?

Deforming objects In deforming an object, we are allowed to do the following

(a) Stretching
(b) Shrinking
(c) Sliding one edge along another.
(d) Pass an object through itself (this move is a complicated one and we'll think about it later)

In deforming an object, we are NOT allowed to

(i) Cut.

(ii) Sew or paste.

Exercise 2 *(1) Using paper, scissors and tape, make two surfaces which can be deformed into one another.*

(2) Draw two surfaces which can be deformed into one another.

(3) Using paper, scissors and tape, make two surfaces which can be transformed into one another by cutting and pasting.

(4) Draw two surfaces which can be transformed into one another by cutting and pasting.

What does topology mean? We are going to think of topology as rubber sheet geometry. That is, we must think of the surfaces we are going to consider as made of a very special kind of rubber sheets, extremely thin (almost without thickness!) and so flexible that we can stretch or shrink them as much as we want, without tearing or losing the "thinness."

For instance, in topology, a very small sheet of paper is the "same" as a huge one, because we can stretch the small one to make it look exactly like the big one.

More generally, two objects are going to be considered "equal" or the "same" if it is possible to deform one into the other (see Figure 3).

(Recall that in the geometry you know, a figure is called a triangle if it has exactly three straight sides. "Having three straight sides"is the main property of the triangle from the *geometric* point of view. We are going to study some of the properties that define objects from the *topological* point of view).

Exercise 3 *(1) Sketch five surfaces which are all the same topologically and all different geometrically.*

(2) Sketch three surfaces which are all the same topologically, all different geometrically and different topologically from the first five objects you've sketched.

(3) Name thee objects from everyday life which are the same topologically but different geometrically.

Remark: There is even a more general way of deform surfaces, imagining our surface in a four dimensional space (but we do not have time to study it, maybe next time...).

Exercise 4 *Consider all the objects of Figure 2 which are surfaces. Which of them are equal from the topological point of view? Group them so that two surfaces are in the same group only if they are topologically the same.*

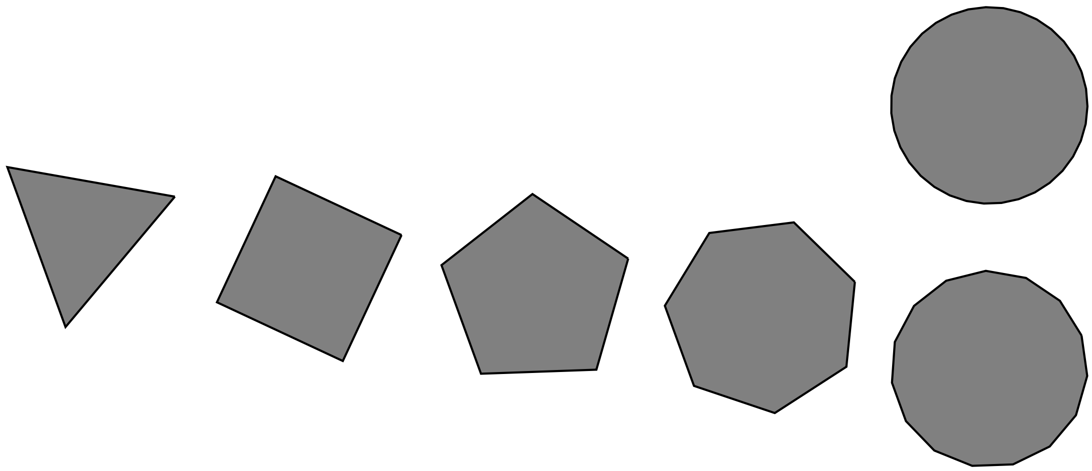


Figure 3: In topology, all these objects are the same!!

0.1 Cylinders and Moebius bands

Experiment 3 Take a sheet of paper and sew two opposite edges to get a cylinder. Is the cylinder the same as the original sheet of paper from the topological point of view? □

Experiment 4 Start with a long strip of paper. Can you glue the two short edges one to each other and obtain a surface which is NOT a cylinder? □

Experiment 5 Start with a long strip of paper. Sew the two short edges to each other. Now take another strip of paper and sew the two shorter edges together to obtain a surface which is different from the first one from the topological point of view. Is it possible to obtain more than two different surfaces in this way? □

Harder question *How many different surfaces can we obtain by sewing the two short edges of a strip of paper?*

Question 3 *Suppose that we perform the previous experiments sewing the long edges instead of the short ones. Will we obtain the same results, from the topological point of view?*

The surface obtained by sewing opposite edges of a strip of papers with one twist is called a *Moebius band*.

Experiment 6 Start drawing a curve on a cylinder, parallel to the edges and halfway between them. Continue drawing it until you arrive where you started. You obtained a *closed curve*.

Do the same with a Moebius band. Can you close the curve? What happens when you come back to the place where you started? How many times must you go around the band before you come back where you started *on the same side of the surface*? □

Remark: Observe that, in the Experiment 6, in order to come back to the same place where you started, on the same side of the surface, one has to go once along the middle curve of the cylinder. On the other hand, one has to go *twice* along the middle curve of the Moebius band to come back to the same place where you started, on the same side of the surface.

Experiment 7 Cut the cylinder and Moebius band along the curve you drew in Experiment 6. What do you get? And if you cut what you got in its middle curve what do you get now? □

Topology is a branch of mathematics which explains phenomena such as the ones we observe in the previous experiments. It is more general than our "rubber sheet geometry" (but it certainly includes it).

0.2 More about Cylinders and Moebius bands

We have seen that if we start with two rectangles and sew an opposite pair of edges without a twist (so the arrows in Figure 4A) point in the same direction) we get a cylinder. And if we sew the edges with a twist (so the arrows in Figure 4B) point in the same direction) we get a Moebius band.

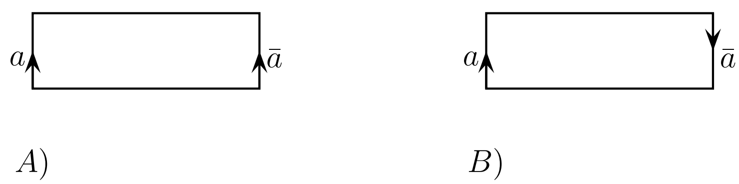


Figure 4: How to obtain a cylinder and a Moebius band

Experiment 8 Start painting a Moebius band with a marker and paint as much as you can without raising the marker from the surface. Do the same with a cylinder and observe the difference. □

Can you list some of the differences between the cylinder and the Moebius band from the topological point of view?

Cylinder	Moebius band

Remark Notice what big difference can be made by the turn of a little arrow.

0.3 The sphere, the torus and the Klein bottle

Experiment 9 Start with three sheets of paper. Sew the edges of each of the sheets as indicated in Figure 5 a) b) and c). (For each of the figures, first glue the pair of edges labeled with $a, \bar{a}$, then the pair labeled with $b, \bar{b}$, and so on until all the labeled edges are sewn. The arrows should point in the same direction where the edges are sewn.)

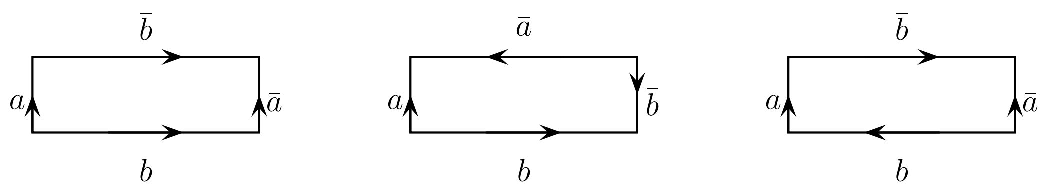


Figure 5: More sewing

What do you get? Can you describe each of these surfaces as the skin of a solid you know? □

Remark The surfaces obtained in the previous experiment by the sewing as in Figure 5A) is called a *torus* and the one from Figure 5B) is a *(topological) sphere.* As you may have realized, the one from Figure 5C) cannot be constructed in our three dimensional space, but it exists in a "mathematical way", and it is called a *Klein bottle.*

Remark We could have used regular letters without a bar to indicate the sewing of Experiment 9 (and sew the pair of edges labeled by exactly the same letter). The reason why we do it with pairs of the form $a, \bar{a}$ and $b, \bar{b}$, etc will appear latter.

Exercise. *Name an object whose skin is a torus.*

The torus we made of paper looks a little collapsed. If we had used flexible rubber band sheets the torus could look as in Figure 6

Experiment 10 Start with two sheets of paper. Cut them and sew the edges of one of the sheets as indicated in Figure 7 a) and the other, as in Figure 7b).

□

Again, if we had used rubber sheets, the surfaces in Figure 7 would look like the surfaces in Figure 8.

Question 4 *Figure 8A) is a torus with one hole. What is Figure 8B)? Can Figure 8B) be deformed into Figure 8C)?*

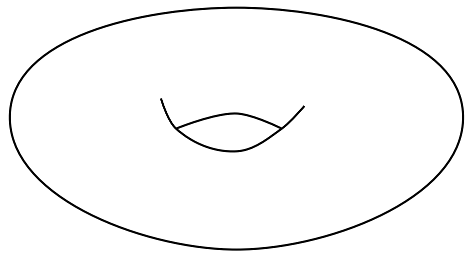


Figure 6: The torus

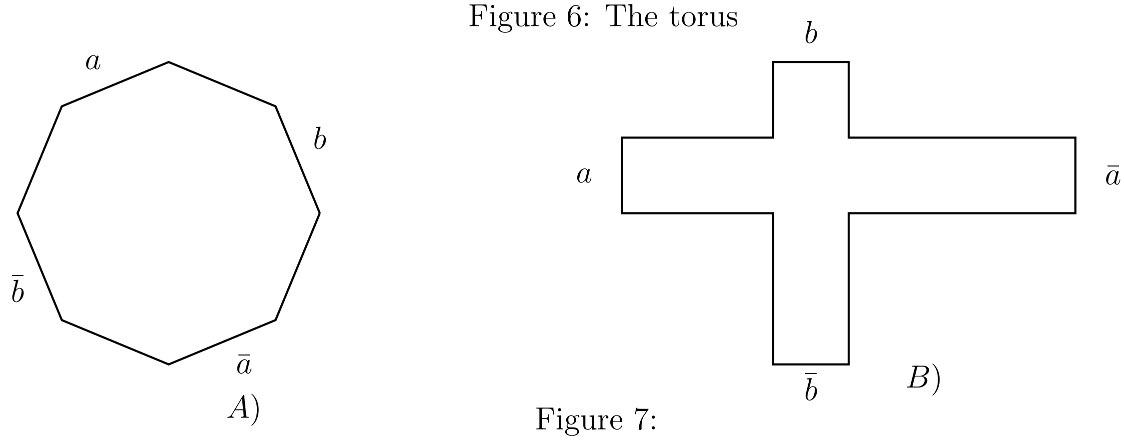


Figure 7:

0.4 Words and surfaces

Let us consider the following set of letters, T. T is an example of something called *alphabet*.

$$T = \{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$$

A *"normal" word in the alphabet* T is an array of letters of T, for instance, $\bar{b}cbb\bar{a}aa\bar{c}$
.

A *circular word in the alphabet* T is an array of a "normal" word written in the letters of T in a ring, as, for instance, in Figure 9 with an arrow telling us in which direction one should read it.

Observe that if we start with the "normal" word $\bar{b}cbb\bar{a}aa\bar{c}$ and we "glue" the beginning to the end we obtain the circular word of Figure 9.

Exercise 5 *Give all the possible normal words in the alphabet T, that arranged in a circle yield the word of Figure 9.*

A *surface word in the alphabet* T is a circular word in which all the letters of T appear exactly once.

Exercise 6 *Give five surface words in the alphabet T.*

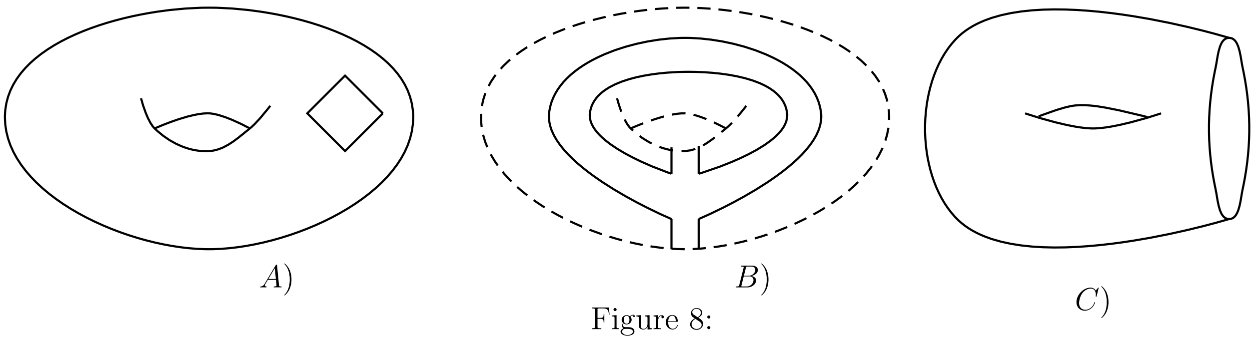


Figure 8:

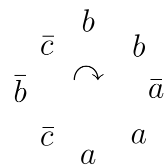


Figure 9: A ring in the letters of $\mathbb{A}_7$

Constructing surfaces using surface words (1) Choose a surface word.

(2) Cut a polygon of twelve edges.
(3) Choose one letter of the surface word and one side of the polygon.
(4) Label the chosen side of the polygon with the chosen letter.
(5) Going clockwise, skip the edge following the one you started with and label the next one with the next letter in the surface word.
(6) Continue like this, labeling alternate edges, until six of the edges are labeled and one can "read" the surface word from the polygon.
(7) Get a strip of paper, and sew one of its short edges to the edge labeled with a and the other, to the edge labeled with $\bar{a}$, with no twisting.
(8) Get a strip of paper, and sew one of its short edges to the edge labeled with b and the other, to the edge labeled with $\bar{b}$, with no twisting.
(9) Get a strip of paper, and sew one of its short edges to the edge labeled with c and the other, to the edge labeled with $\bar{c}$, with no twisting.

Exercise 7 *What surface word will produce a cylinder? What surface words will produce a torus with one hole?*

Experiment 11 For each pair of normal words, make the corresponding circular words and construct the associated surface and decide if they are topologically the same.

(1) $a\bar{a}b\bar{b}c\bar{c}$, $a\bar{c}c\bar{a}b\bar{b}$
(2) $abc\bar{a}\bar{b}\bar{c}$, $ab\bar{a}\bar{b}c\bar{c}$

What is the name of each of the constructed surfaces? □

Exercise 8 *Give a pair of different surface words that yield the same surface from the topological point of view.*

0.5 Constructing new surfaces from old surfaces

Exercise 9 *Complete Table 1*

Surface	Number of holes	Number of sides	Other description
Sphere			
Disk			
Cylinder			
Moebius band			
Klein bottle			
Torus			
$ab\bar{a}\bar{b}$			
$abc\bar{a}\bar{b}\bar{c}$			

Table 1: "Adding" surfaces

Question 5 *We know two properties of the surfaces: number of holes and number of sides. Are those two properties enough to* characterize *the surfaces? In other words, if two surfaces have the same number of holes and the same number of sides, are they topologically equivalent?*

A recipe for constructing surfaces from other surfaces *(see Figure 10)*

(1) Start with two surfaces.
(2) Cut a hole on each of them
(3) Sew the two surfaces along the hole.

Exercise 10 *Complete Table 2.*

The number of holes of a surface is a feature that does not change when we deform the surface. Another number that does not change is the number of *handles* or tori (plural of torus) one has to sew together to obtain the surface.

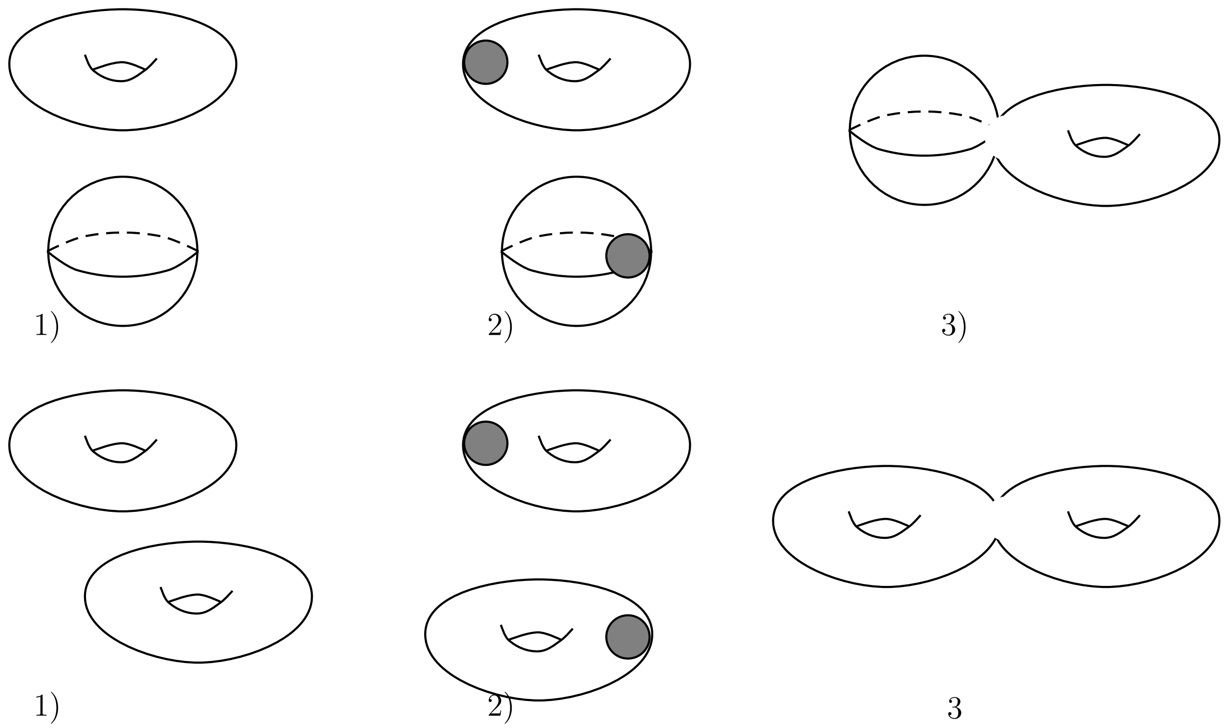


Figure 10: Two examples of the new recipe

0.6 Deformation classes of surfaces

In our first class, we said that two surfaces are "equal" if they can deformed into one another. To be more precise, one should say that two surfaces that can be deformed into one another are *equivalent*. Hence, "can be deformed into one another" defines an equivalence relation on the set of all surfaces. Two surfaces that are equivalent are said to be in the same *equivalence class*.

Very Important Theorem *All the surfaces in one of these equivalence classes have the same number of sides (they all have two sides or one side), the same number of handles and the same number of holes. Reciprocally, if two surfaces both have two sides or one side, they have the same number of handles and the same number of boundary components, then they are in the same equivalence class.*

Therefore, number of sides, number of handles and number of holes characterize *the equivalence classes of surfaces.*

Translation of the Very Important Theorem to technical math languange *All the surfaces in one of these equivalence classes have the same orientability property*

Surface	Sphere	Disk	Cylinder	Torus	Torus with one hole
Sphere					
Disk					
Cyllinder					
Torus					
Torus with hole					

Table 2: Which surface does one obtain by sewing pairs of surfaces?

(they are all orientable or all non-orientable), the same genus and the same number of boundary components. Reciprocally, if two surfaces are both orientable or both non-orientable, and they have the same genus and the same number of boundary components, then they are in the same equivalence class.

Therefore, orientability, genus and number of boundary components characterize *the equivalence classes of surfaces.*

Exercise 11 *Complete the English-math dictionary*

(1) *Number of sides:*
(2) *Having one side:*
(3) *Having two sides:*
(4) *: boundary component*
(5) *: genus*

Remark 12 We have not given a rigourous proof of the above theorem in the way that actual mathematicians do (we would need many more classes for that). Nevertheless, we gave some of the "first steps" of a proof, which convinces our intuition that the result is true. □

From now on, we are going to study only surfaces with two sides.

handles holes	0	1	2	3	4
0					
1					
2					
3					

Table 3: All surfaces with at most 4 handles and at most 3 holes

Exercise 12 *Complete the Table 3 with figures of two-sided surfaces.*

Remark 13 Table 3 shows the beginning of a complete classification of two-sided surfaces with at most four handles and at most three holes. By continuing the table, we obtain a complete classification of all two sided surfaces. That is, any two-sided surface we can think of, can be deformed like one of the squares of the table. □

Question 6 *In Table 3 we list some of the two-sided surfaces. Which of the surfaces of Table 3 can be the outer layer of a solid?*

Can a one-sided surface be the outer layer of a solid?

More generally, which surfaces can be the outer layer of a solid?

1 Curves - a preview

From the Clay Institute webpage *If we stretch a rubber band around the surface of an apple, then we can shrink it down to a point by moving it slowly, without tearing it and without allowing it to leave the surface. On the other hand, if we imagine that the same rubber band has somehow been stretched in the appropriate direction around a doughnut, then there is no way of shrinking it to a point without breaking either the rubber band or the doughnut. We say the surface of the apple is "simply connected," but that the surface of the doughnut is not. Poincaré, almost a hundred years ago, knew that a two dimensional sphere is essentially characterized by this property of simple connectivity, and asked the corresponding question for the three dimensional sphere (the set of points in four dimensional space at unit distance from the origin). This question turned out to be extraordinarily difficult, and mathematicians have been struggling with it ever since.*

An old recipe to get a million dollars *Find a proof of the Poincaré conjecture.*This recipe is now outdated.Perelman posted two short papers in 2002 and 2003, in which he proved a result that implies this conjecture

2 Deformation classes of curves in a surface

A *curve on a surface* is a (topological) circle on that surface with an *orientation*. In other words, it is a closed path with an arrow which tell us in which direction to walk along it. (By *closed* we mean the beginning and end of the path are glued together; the *orientation* is given by the arrow).

Our next goal is to understand better the set of all curves on a (fixed) surface. As we did with surfaces, we are going *classify* them, that is, we are going to divide the set of all curves on a surface into equivalence classes.

Again as we did with surfaces, we are going to think that the curves are made of extremely flexible rubber bands. We can shrink, expand, slide curves along the surface. We are not allowed to break them or separate them from the surface.

Two curves on the same surface are *equivalent* (or, the "same" from the topological point of view) if one can be deformed into the other.

Exercise 13 *Make a cylinder with paper and draw five different curves on it.*

Question 7 *How many deformation classes of curves do the following surfaces have?*

(1) The sphere
(2) A rectangle (or sphere with one boundary component)
(3) The cylinder (or sphere with two boundary components)
(4) The torus
(5) A surface of genus five with four boundary components

Choose different deformation classes of curves on the cylinder and draw a curve for each of these classes. Can you give a label for each deformation class of curves in the cylinder?

From now on, we are going to choose a surface and study the deformation classes of curves on a fixed surface. The surface we are going to choose is the one constructed by the rules given in the previous notes, using the circular word obtained by gluing both ends of the normal word $abABcC$.

Remark 14 Instead of choosing the surface constructed with the word $abABcC$, we could have chosen any surface and made this study. In most cases, using a different surface, one obtains a different set of deformation classes.

On the other hand, two surfaces that can be deformed one into the other, have the "same" set of deformation classes of curves. By "the same" we do not mean *exactly* the same (after all, we are considering curves on different surfaces). We mean that there is a one to one correspondence between these two sets of equivalence classes. This correspondence is "deep": for instance, if a pair of classes of one surface "behaves" in a certain way, then the corresponding pair of classes of the other surfaces will "behave" in the same way. □

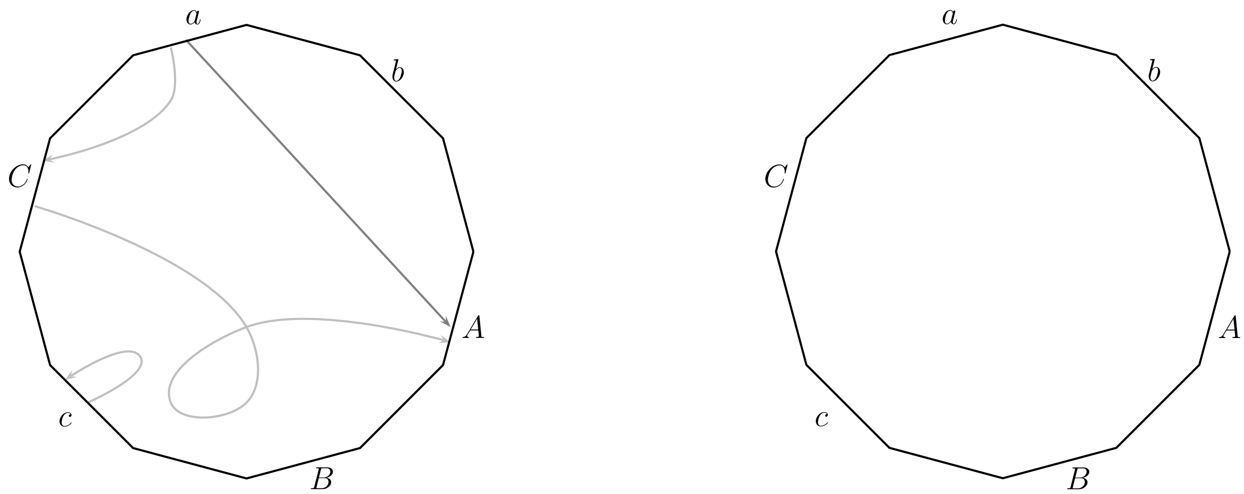
Remark 15 Note that **a surface does not intersect itself** (even if we can make it pass through itself to deform it, the final deformed surface cannot intersect itself). The glass model of the Klein bottle shown in class is not a real Klein bottle. A real Klein bottle cannot be constructed in our three dimensional space. The model is just a *trick* to help us visualize it.

On the other hand, **a curve on a surface may intersect itself**.

□

Exercise 14 *Look at the path in dark gray in Figure 11 i). After gluing together the edges of the polygon as indicated by the labels, the path becomes a closed oriented curve (observe that it has an arrow)*

Analogously, the two pieces of light gray paths on Figure 11 i) make a curve after gluing together the labeled edges. Hence, we have two curves. Are both curves in the same deformation class?



$i)$ $ii)$

Figure 11: Curves on the surface $abABcC$

The two curves of Figure 11 $i)$ are equivalent. The darker curve is shorter and simpler, and so, easier to deal with. Hence, when we try to analyze a deformation class of curves, we are going to consider the "simplest" curves in the deformation class.

The shortest curve in Figure 11 $i)$ "enters" through the edge labeled with a and "leaves" through the edge labeled by A. We will denote the deformation class of this curve by the circular word a.

Exercise 15 *In Figure 11 $ii)$ draw*

(1) the shortest curve of the deformation class b.

(2) the shortest curve of the class deformation class B.

(3) a curve in the deformation class b which is not the shortest.

Exercise 16 *Observe that the two paths in Figure 12 $i)$ form a curve. Draw one of the simplest curves Figure 12 $i)$ which can be deformed into the curve we have there. The new curve has two pieces. For each of the pieces, say where they enter and where they come out.*

We will denote the deformation class of this curve by the circular word BA.

Remark 16 We are using circular words for two different purposes that should not be confused: to construct surfaces and to label deformation classes of curves on a fixed surface. □

Exercise 17 *(1) Draw in Figure 12 $iii)$ one of the simplest curve of the class labeled by the circular word Bc.*

(2) Draw in Figure 12 $iv)$ one of the simplest curve of the class aBc.

(3) Write two cyclic words of at least four letters and draw the corresponding shortest curves in Figure 12 $v)$ and $vi)$.

Exercise 18 *Draw in Figure 13 i) one of the simplest curves of the class* bBc*. Study your drawing. Can you find a curve in the same deformation class which is shorter than the one you drew?*

A circular word in which none of the pairs aA, Aa, bB, Bb, cC or Cc appear is called a *reduced word.*

Exercise 19 *Write two reduced words of seven letters.*

Exercise 20 *Draw in Figure 13 ii) one of the simplest curves of the class* $ABBc$*. Does the curve you draw intersect itself? Is it possible to avoid those intersections?*

Exercise 21 *Draw in Figure 13 iii) one of the simplest curves of the class* $ABCBc$*. Does the curve you draw intersect itself? Is it possible to avoid those intersections?*

Exercise 22 *Draw in Figure 13 iv) one of the simplest curves of the class* $AABaBc$*. Does the curve you draw intersect itself? Is it possible to avoid those intersections?*

Question 8 *What do the words of the three previous exercises have in common? (Hint: The words* $ABcccdcdcDcccDBc$*,* $aaabABaabcadaBcccc$ *and* $ABccBc$ *share the same characteristic)*

Exercise 23 *Complete the following sentence:*
If a cyclic reduced word ... then any curve labeled by the word ...

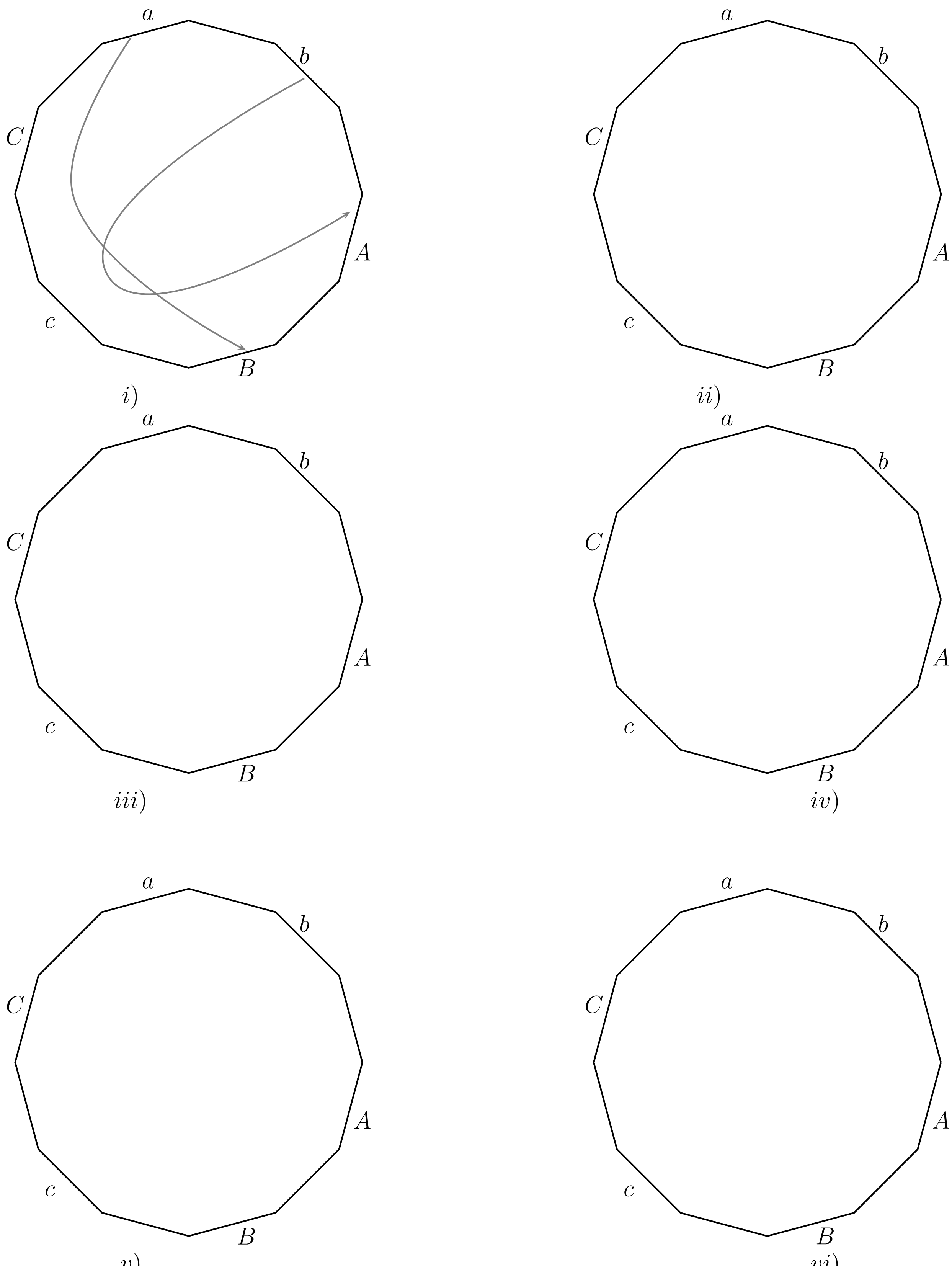


Figure 12: More curves on the surface $abABcC$

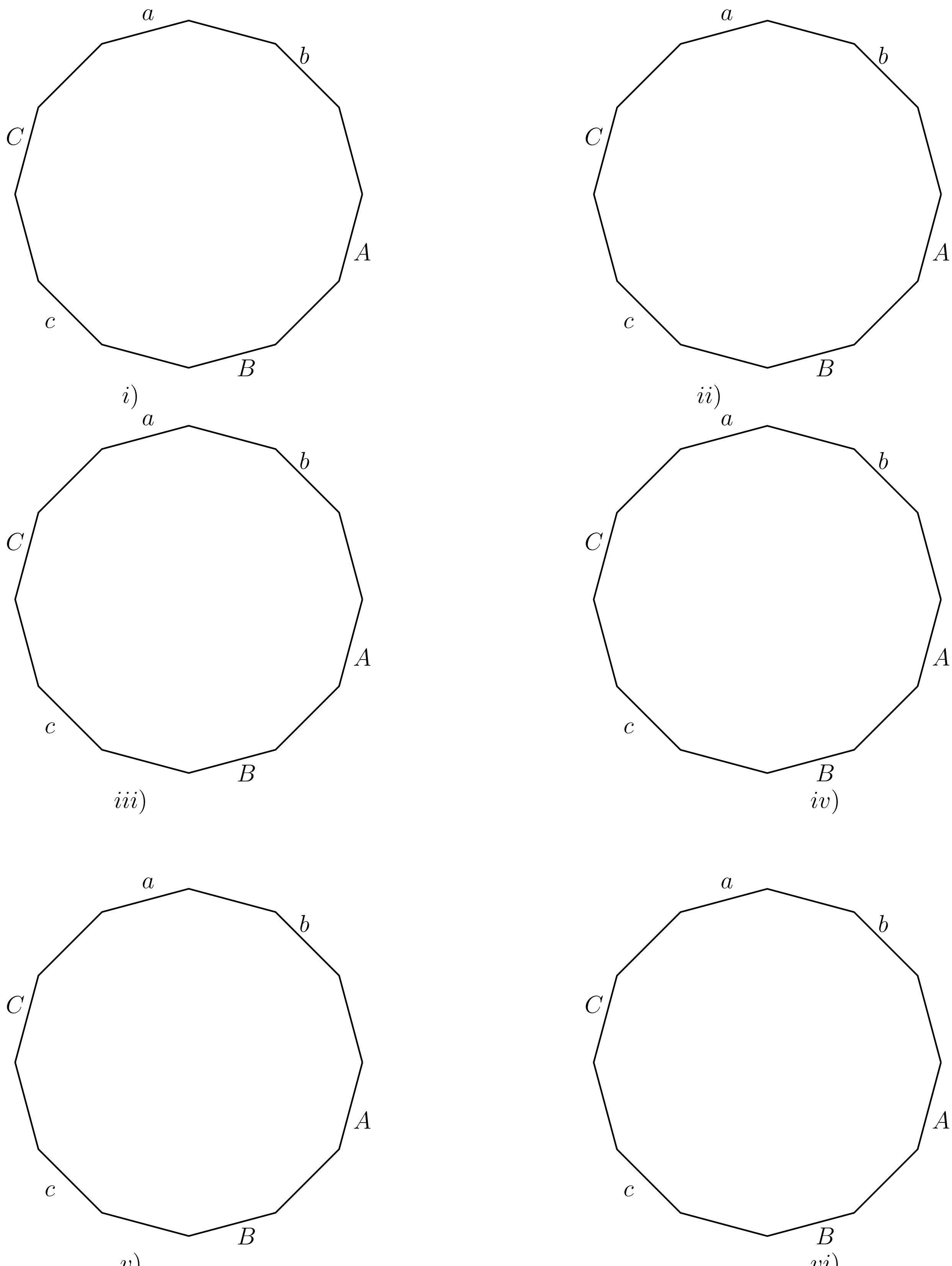


Figure 13: Even more curves on the surface $abABcC$