THURSTON'S EXTRAORDINARY CURVE

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Among all closed curves on a surface, there is a subset that deserves special mention: the curves with no crossings or *simple curves*. At the end of the XIX century, 1895 Poincaré published "Analysis situs", starting topology as a new area of Mathematics [3, Translator's Introduction]. Between 1899 and 1904, he published "Complé ments à l'Analysis situs" [3] correcting mistakes and tying up loose ends. In the fifth of these complements, he discussed the first characterization of simple closed curves, namely, a curve is simple if an only if all of the lifts to the universal cover are pairwise disjoint.

Despite their name, a simple curve can be far from simple (if we understand simple as opposite to complex), see for instance, Figure 2. This curve was constructed by Bill Thurston in 1971. One starts with a flat disk of uncooked dough with three wooden rods standing in the dough. Label these 1, 2 and 3. Now, make a very thin closed curve with food coloring that goes tightly around rods 2 and 3. Second, perform the following two operations over and over a few times to get the curve of Figure 2. The first operation is interchange rods 1 and 2 by moving them halfway around a loop in the counterclockwise direction. Then interchange 2 and 3 in a circular clockwise direction.

Here is an excerpt of a story written by Dennis Sullivan about the curve in Figure 2: A couple of days later the Berkeley grad students invited me to join the painting math frescoes on the corridor wall separating their offices from the elevator foyer. While milling around before painting a grad student (Thurston) came up to ask "Do you think this is interesting to paint?" It was a complicated smooth one dimensional object encircling three points in the plane. I asked "What is it?" and was astonished to hear "It is a simple closed curve". I said "You bet it's interesting!". So we proceeded to spend several hours painting this curve on the wall. It was a great learning and bonding experience. For such a curve to look good it has to be drawn in sections of short parallel slightly curved strands (like the flow boxes of a foliation) which are subsequently smoothly spliced together. When I asked how he got such curves , he said by successively applying to a given simple curve a pair of Dehn twists along intersecting curves. The "wall curve painting", two meters high and four meters wide, dated and signed, lasted on that Berkeley wall with periodic restoration for almost four decades before finally being painted over a few years ago. (See also [2]).

For more discussion about this curve see [1] .

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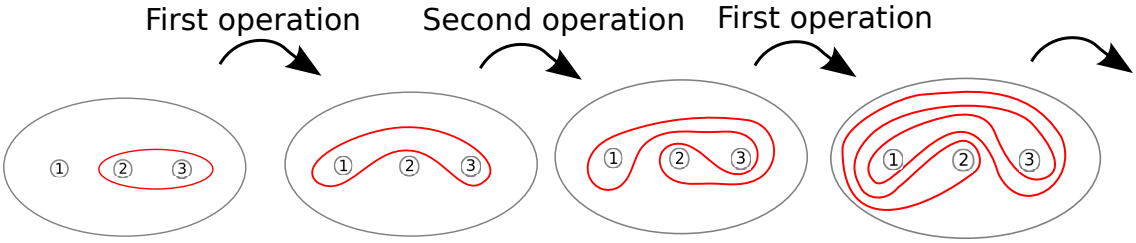

FIGURE 1. Building Thurston's simple curve

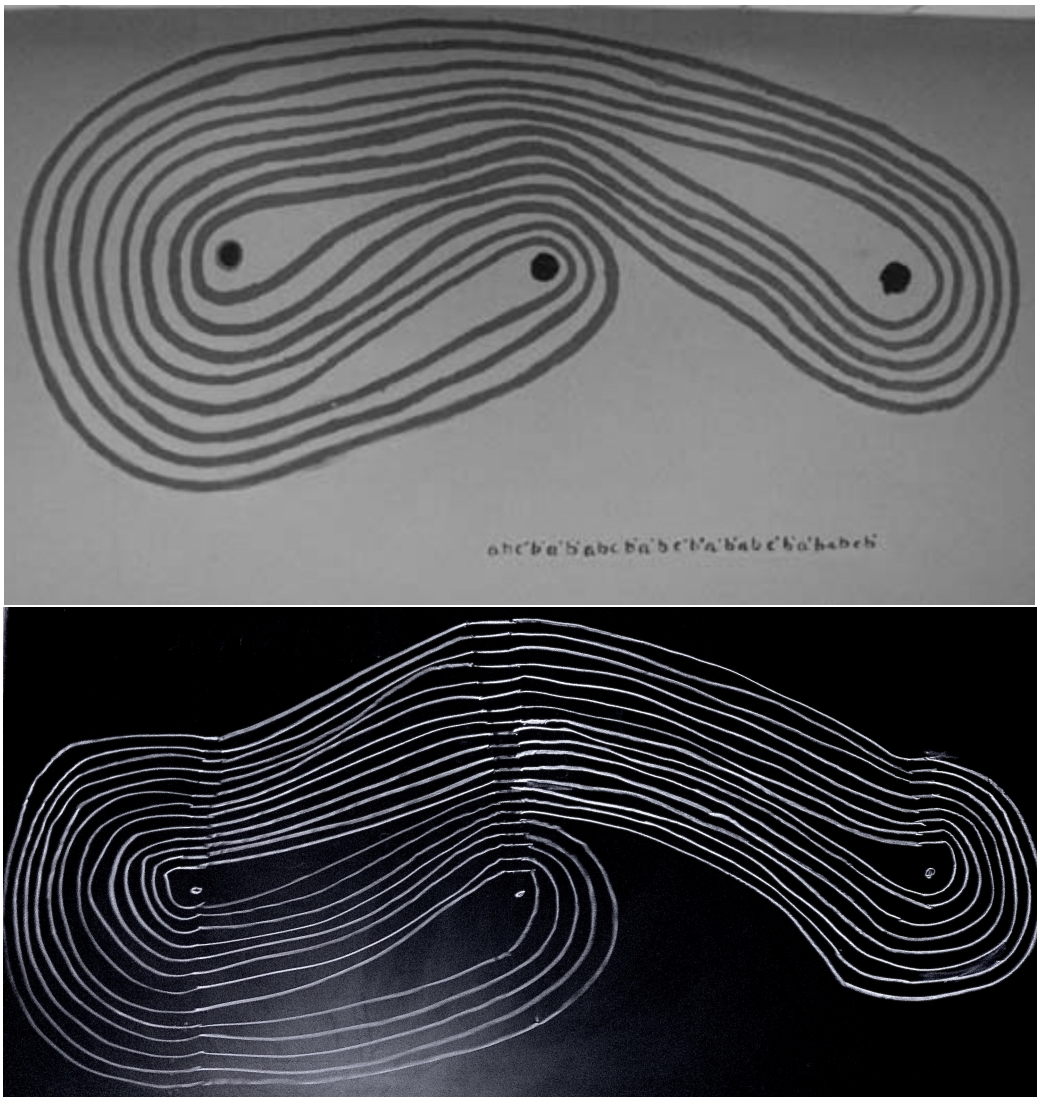

FIGURE 2. The painting on the wall the iterations of Thurston simple curve by Thurston and Sullivan (top) and a picture of further iterations of the Thurston curve by IISER Ph.D. students

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