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The Goldman bracket and the intersection of curves on surfaces

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Abstract. In this note, we discuss a Lie algebra structure of Goldman from an elementary point of view, together with its relation to the structure of intersection and self-intersection of curves on surfaces. We also list examples and mention some of the open problems in the area.

This Lie algebra is defined by combining two well known operations on homotopy classes of curves: the transversal intersection and the composition of directed loops which start and end at the same point.The Lie algebra turns out to be a powerful tool and its structure still contains many mysteries.

1. Introduction

In the eighties, Goldman [**10**] described an unexpected Lie algebra structure on linear combinations of free homotopy classes of directed closed curves on an orientable surface. This Lie algebra is defined by combining two well known operations on curves: the transversal intersection and the composition of directed loops which start and end at the same point. The Lie algebra turns out to be a powerful tool and its structure still contains many mysteries. It has been generalized in diverse directions (see, for a sample, Section 6), has posed interesting questions (see Section 6) and motivated a great deal of research.

The goal of these notes is to discuss this Lie algebra from an elementary point of view, starting with a discussion of the linear space where the bracket is defined (Section 2), followed by the precise definition (Section 3), continuing with examples (Section 4) and a study of the relation between the bracket and intersection of curves (Section 5) and ending with some of the open problems and further development in this area (Section 6).

2. The $\mathbb{Z}$-module of curves

Fix a surface Σ, with or without boundary, and possibly not compact. Consider two closed oriented curves a and b on Σ, that is, two maps a and b from the oriented circle to Σ. The curves a and b are said to be *freely homotopic* if there exists a map from a cylinder C to Σ such that the restriction of F to one of the

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(oriented) boundary components of C coincides with a and the restriction to the other, coincides with b.

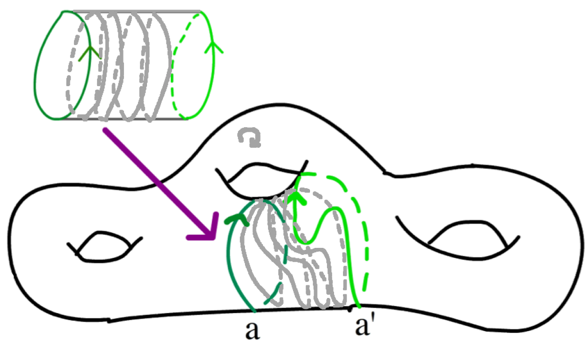


FIGURE 1. A (free) homotopy between the curves a and a'

Lower case letters $a, b, c, \ldots$ will be used to denote curves; capital case letters $A, B, C \ldots$, will denote free homotopy classes of curves. The class of the trivial loop is denoted by $\circ$. If x is a curve, $\mathcal{FH}(x)$ denotes its free homotopy class and $\bar{x}$ the curve x with its direction reversed. Also, for each positive integer n, x^n is the curve that goes n times around x in the same direction.

Denote by π_0 the **set** of free homotopy classes of Σ (see Remark 6.5 for a discussion of this notation). The boldface on "set" is meant to emphasize the fact that a priori there is not an obvious algebraic structure on π_0, as opposed to Poincarè's celebrated group structure on the set of (based) homotopy classes of closed curves in a space.

The free module of linear combinations over the ring of integers with basis π_0 is denoted $\mathbb{Z}[\pi_0]$. Thus an element of $\mathbb{Z}[\pi_0]$ is a formal linear expression of free homotopy classes with integer coefficients. An example of such linear combination using the above notation is

$$3\mathcal{FH}(a) - \mathcal{FH}(\bar{a}) + \circ + 7B + \mathcal{FH}(a^3).$$

Note that the class of the trivial loop is not zero; in symbols, $\circ \neq 0$. Also, if a is any curve then $\mathcal{FH}(\bar{a}) \neq -\mathcal{FH}(a)$ and $\mathcal{FH}(a^3) \neq 3\mathcal{FH}(a)$. Moreover, $\circ$, $\mathcal{FH}(a)$, $\mathcal{FH}(\bar{a})$ and $\mathcal{FH}(a^3)$ are all different members of the basis π_0.

EXERCISE 1. *Show that there is a natural bijection between π_0 and the set of components of the space of maps from the circle to Σ, with the compact-open topology. (This is the reason why the set of free homotopy classes is denoted by π_0.)*

EXERCISE 2. *Let X be a path connected space and consider $x_0 \in X$. Prove there is a bijection between the set of free homotopy classes of maps of from the oriented circle to X and the set of conjugacy classes of elements of $\pi_1(X, x_0)$, the fundamental group of X with base point x_0. (Hint: Show that there is a map from $\pi_1(X, x_0)$ to the set of free homotopy classes of maps from the circle to X, which is constant on conjugacy classes).*

COROLLARY 2.1. *If Σ is a connected surface then there is is a bijection between the set of free homotopy classes of closed directed curves on Σ and the set of conjugacy classes of $\pi_1(\Sigma, x_0)$.*

If Σ is an orientable surface with non-empty boundary then the fundamental group of Σ is free. Consider a minimal set $\{x_1, x_2, \ldots, x_n\}$ of generators of this free group. A *cyclic word* is an equivalence class of words in $\{x_1, x_2, \ldots, x_n\}$ and their inverses related by a cyclic permutation of their letters. A word is *reduced* if it does not contain the juxtaposition of a generator and its inverse.

EXERCISE 3. *Show that there is a natural one-to-one correspondence between cyclic reduced words and free homotopy classes of curves on a connected, orientable surface with boundary* Σ. *(The empty word is considered to be a cyclic reduced word. It corresponds to the conjugacy class of the identity of the fundamental group.)*

By Exercise 3, if Σ is a surface with boundary, we can think of $\mathbb{Z}[\pi_0]$ as the free $\mathbb{Z}$-module with basis the cyclic reduced words in a (minimal) set of generators of $\pi_1(\Sigma, x_0)$ and their inverses.

3. The Goldman bracket

From now on, we will assume that the surface Σ is orientable and has a chosen orientation. Our next goal is to define a Lie bracket $[\cdot,\cdot]$ on $\mathbb{Z}[\pi_0]$. Recall that a Lie bracket is a bilinear map in two arguments, which is skew-symmetric and which satisfies the three term Jacobi identity (that is, for every triple α, β, γ in the module, $[\alpha, [\beta, \gamma]] + [\gamma, [\alpha, \beta]] + [\beta, [\gamma, \alpha]] = 0$. We define this bracket for each pair of elements of the basis π_0, and then, extend it to $\mathbb{Z}[\pi_0]$ by bilinearity.

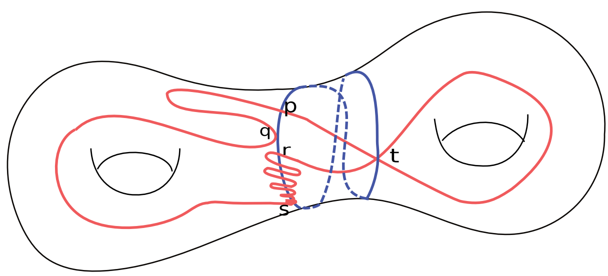


FIGURE 2. Points p and r are transversal double points. Points q and s are not transversal. Point t is transversal and not double, but triple

Consider two free homotopy classes A and B and a pair of representatives a and b. We can assume (by performing a small homotopy if necessary) that a and b are representatives of A and B intersecting only in transversal double points $p_1, p_2, \ldots, p_n$ (see Figure 2 for examples of transversal double points.) There are finitely many intersection points because of transversality.

For each $i \in \{1, 2, \ldots, n\}$, denote by $a \cdot_{p_i} b$ the loop that starts at p_i, goes around a until reaching p_i again, and then "turns" and goes around b. (see Figure 3). That is, $a \cdot_{p_i} b$ is the based loop product of a and b with base point p_i.

Each intersection point p_i determines an orientation, given by going from the positive branch of a to the positive branch of b. We associate a sign s_i to p_i, by setting $s_i = 1$ if the orientation of the surface Σ coincides with the orientation determined by p_i and $s_i = -1$ otherwise.

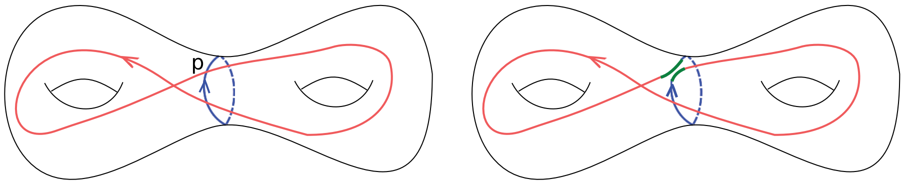


FIGURE 3. Based loop product (right) at the intersection point p of the curves on the left.

Finally, the Goldman bracket $[A, B]$ is defined as the sum over the set of intersection points of a and b, of the signed free homotopy classes of the loop product of a and b at p_i. In symbols,

$$[A, B] = \sum_{p_i \in a \cap b} s_i \cdot \mathcal{FH}(a \cdot_{p_i} b)$$

Since the bracket was defined using representatives, one is required to check that it is, after collecting terms, independent of the choices.

THEOREM 3.1 ([**10**]). *The Goldman bracket is well defined, skew symmetric and satisfies the Jacobi identity. In symbols, for each triple of elements* $\alpha, \beta, \gamma \in \mathbb{Z}[\pi_0]$,

$$[\alpha, \beta] = -[\beta, \alpha] \text{ and } [\alpha, [\beta, \gamma]] + [\gamma, [\alpha, \beta]] + [\beta, [\gamma, \alpha]] = 0.$$

PROOF. We give here a rough sketch of the proof that the bracket is well defined and we refer the reader to [**10**] for a more precise proof: Given two pairs of curves a and b, and a' and b', with a homotopic to a' and b homotopic to b', there exists a homotopy deforming simultaneously one pair of curves to the other pair. For a typical homotopy there are finitely many times during this deformation $(a, b) = (a_0, b_0)$, $(a_1, b_1), \ldots, (a_n, b_n) = (a', b')$ such that the difference in the intersection points between (a_i, b_i) and (a_{i+1}, b_{i+1}) can be described by the Figures 4 and 5. Since by Exercises 5 and 6 the bracket is unchanged when replacing each figure by the corresponding one, then the bracket does not change through the homotopy. $\square$

EXERCISE 4. *Prove that the Goldman bracket satisfies the Jacobi identity. (Hint: Start with a concrete example of three of curves a, b, c intersecting pairwise possibly in more than one point, and show that the three triple brackets give terms that cancel in pairs. Then generalize your argument to all classes.)*

EXERCISE 5. *Let a and b be two curves that intersect in only in transversal double points and let p and q be two of these intersection points. Suppose that p and q lie on the surface as in Figure 4, left. (This configuration of arcs is called a bigon). Show the terms of the Goldman bracket of $\mathcal{FH}(a)$ and $\mathcal{FH}(b)$ corresponding to p and q cancel. (Make sure to consider all possible directions of the curves a and b).*

EXERCISE 6. *Let a and b be two curves that intersect only in transversal double points that locally near some point are made from the branches of either the left side or the right side of Figure 5. Note in each case one curve uses two branches and the other a single branch. The passage from left to right is called a triple point move. Treat all of cases and show the bracket is unaffected by a triple point move. (Thus it must be shown that the terms of the bracket corresponding to the intersection points p and q, are equal to the terms of the bracket corresponding to the points r and s.)*

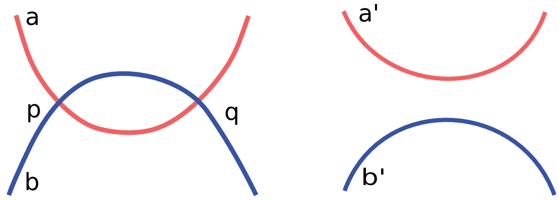


FIGURE 4. Exercise 5

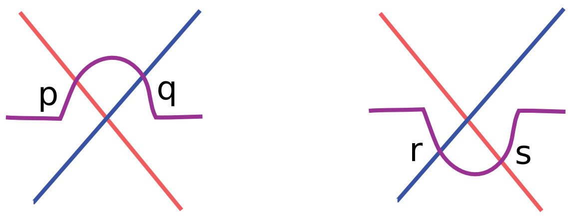


FIGURE 5. Exercise 6

As said in the beginning, the set π_0 of free homotopy class does not have an obvious algebraic structure. In particular, curves with no base point cannot be multiplied, (note that loops with the same base point can be multiplied). When computing the bracket, one considers all points where it is possibly to multiply (the intersection points), and in each of these points one performs the multiplication. Then the linear combination of outcomes of the multiplication yields the Goldman bracket.

EXERCISE 7. *Recall that the fundamental group of the torus is a free abelian group in two generators, a and b. Prove the following:*

(1) There is a bijection between π_0 and the set $\{a^i b^j; i, j \in \mathbb{Z}\}$.
(2) Compute $[a^i b^j, a^h b^k]$ for each $i, j, h, k \in \mathbb{Z}$. (Hint: Compute $[a, b]$, $[a^i, b]$, $[a^i, b^k]$ and $[a^i b^j, b^k]$. The formula for the general case involves a determinant)
(3) Prove algebraically that the Goldman bracket for the torus satisfies the Jacobi identity.

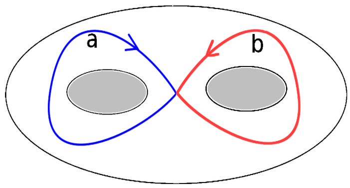
EXERCISE 8. *Prove that the Goldman Lie algebra on a (non-necessarily connected) oriented surface Σ is the direct sum of the Lie algebras for the connected components of Σ.*

4. Examples

Assume that the surface Σ is connected. Since by Exercise 2, there is a bijection between π_0 and the set of conjugacy classes of $\pi_1(\Sigma)$, we can (and will) identify π_0 with the set of conjugacy classes.

If $\alpha \in \pi_1(\Sigma)$, denote by $c(\alpha)$ the conjugacy class of α.

EXERCISE 9. *Show that the bracket of the two classes represented by the two curves on the left of Figure 3 has two terms that do not cancel. (This exercise is not easy. It can be done by using the fact that the fundamental group of the surface can be described as a free product with amalgamation).*



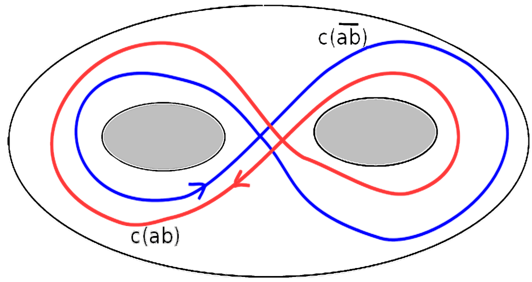


FIGURE 6. A set of generators of the fundamental group of the pair of pants (left) and representatives of the classes in Exercise 10(3) (right)

EXERCISE 10. *Consider the pair of pants with standard generators a and b as in Figure 6. Show that for one of the two possible orientations of the pair of pants, the following holds:*

(1) $[c(ab), c(\bar{b}\bar{a})]) = c(ba\bar{b}\bar{a}) - c(ab\bar{a}\bar{b})$.
(2) $[c(ab), c(aab)]) = 0$.
(3) $[c(ababab), c(ab)]) = 3(c(bababbaa) - c(ababaabb))$.

We showed in Exercise 3 that for a connected surface with boundary, there is a one-to-one correspondence between π_0 and the cyclic reduced words in a minimal set of generators of the fundamental group. Thus, the Goldman bracket determines a Lie algebra on the module generated by cyclic reduced words in a finite alphabet. This bracket was described in a purely combinatorial way in [**2**]. Here is a rough idea of this: The bracket of two cyclic reduced words is computed by finding certain "cuts" or spaces between letters of each word (each pair of cuts corresponding to a minimal intersection point), open each word up obtaining two linear words, and take the cyclic reduced word determined by the concatenation of these two words. Exercise 10 illustrates these ideas.

5. Relation between the bracket of two curves and the number of intersection points.

Recall that the *geometric intersection number* $\mathrm{i}(A, B)$ *(or just intersection of two free homotopy classes A and B)* is the smallest number of mutual transversal intersection points of pairs of representatives a and b, counted with multiplicity. Equivalently, the minimal intersection number can be defined as the smallest number of mutual intersection points of pairs of representatives, provided the considered pairs intersect only in transversal double points.

The *self-intersection of a free homotopy class A,* $\mathrm{SI}(A)$ is the smallest number of transversal crossings of representatives of A, counted with multiplicity.

The result of Hass and Scott [**11**] rephrased below in Theorem 5.1 will be used to determine minimal self-intersection and intersection numbers of free homotopy classes.

THEOREM 5.1 ([**11**]). *If a is a closed curve on an orientable surface Σ, and the self-intersection number of a is strictly larger than the minimal self-intersection of its free homotopy class then one of the following holds:*

(1) there are disjoint arcs X and Y of the circle (parametrizing the map a) such that a identifies the endpoints of X and Y and $a|_{X\cup Y}$ defines a null-homotopic loop on Σ.

(2) there is a sub arc X of the circle such that a identifies the endpoints of X and $a|_X$ defines a null-homotopic loop on Σ.

An element of $\mathbb{Z}[\pi_0]$ can be written as a linear combination $c_1A_1+c_2A_2+\cdots+c_nA_n$, where each $A_i \in \pi_0$ and $A_i \neq A_j$ if $i \neq j$. The *Manhattan norm (or l_1-norm) of* $c_1A_1+c_2A_2+\cdots+c_nA_n$, $M(c_1A_1+c_2A_2+\cdots+c_nA_n)$ is the number of terms counted with multiplicity, that is, $|c_1|+c_2|+\cdots+|c_n|$.

Since the Goldman bracket of two classes A and B is defined as a sum over the intersection points of representatives, the number of terms cannot exceed the intersection number $\mathrm{i}(A,B)$. Indeed, one can compute the bracket as a sum over the intersection points of two representatives that intersect minimally, thus there at are at most $\mathrm{i}(A,B)$ terms. After "assigning" a term to each intersection point, and taking the algebraic sum, there might be some cancellation in the collecting process. Hence we have,

PROPOSITION 5.2. *For each pair A, B of free homotopy classes, $M[A,B] \leq \mathrm{i}(A,B)$.*

EXERCISE 11. *Let a and b be as in Exercise* 10*. Prove the following. (Recall that* SI *denotes self-intersection.)*

(1) $M[c(ab),c(aab)] < \mathrm{i}(c(ab),c(aab))$.

(2) $M[c(ab),c(\bar{a}\bar{b}))] = \mathrm{i}(c(ab),c(\bar{a}\bar{b}))$.

(3) $M[c(ab),c((ab)^3)] = 2\cdot 3\cdot \mathrm{SI}(ab)$.

(4) $M(C,D) = i(C,D)$ where C and D are the free homotopy classes of the two curves in Figure 3*, left.*

(Hint: Show using Theorem 5.1*, $\mathrm{SI}(c(ab)) = 1$, $\mathrm{i}(c(ab),c(aab)) = 2$ and $\mathrm{i}(c(ab), c(\bar{a}\bar{b})) = 2$.)*

By Exercise 11(1), $M(A,B)$ and $i(A,B)$ are not always equal and by Exercise 11(2), they are not always distinct. A natural question is whether there are necessary conditions on A and B so that the equality $M[A,B] = i(A,B)$ holds. The first answer to this question was given by Goldman who proved:

THEOREM 5.3 ([**10**]). *If A has a representative with no self-intersection and $[A,B]=0$ then A and B have disjoint representatives. In other words, $[A,B]=0$ if and only if $i(A,B)=0$, provided that $\mathrm{SI}(A)=0$. Rephrased in our notation, if $\mathrm{SI}(A)=0$ and $M[A,B]=0$ then $i(A,B)=0$.*

We generalized Goldman's result as follows:

THEOREM 5.4 ([**3**]). *If A has a representative with no self-intersection then the Manhattan norm of the bracket of A and B equals the intersection number of A and B. In symbols, if $\mathrm{SI}(A)=0$ then $M[A,B]=i(A,B)$.*

Exercise 11(4) illustrates Theorem 5.4.

The main tool in the proof of Theorem 5.4 is to write the fundamental group of Σ as the free product with amalgamation of the components of $\Sigma \setminus a$ if a separates

Σ or the HNN extension of $\Sigma \setminus a$ if a does not separate. Since a is simple, conjugacy classes and the terms of the bracket can be described combinatorially in terms of these structures.

Jointly with Krongold we proved:

THEOREM 5.5 ([**5**]). *If Σ is a surface with non-empty boundary, and A is a free homotopy class of curves in A, which is not a proper power, then the Manhattan norm of the Goldman bracket of A^p and A^q, $M[A^p, A^q]$ is $2 \cdot p \cdot q$ times the minimal possible number of self-intersections of representatives of the free homotopy class A, provided that p are q distinct positive integers and either p or q is larger than three.*

Exercise 11(3) illustrates Theorem 5.5.

EXERCISE 12. *Show that for each $A \in \pi_0$, the terms of the bracket $[A, \bar{A}]$ are conjugacy classes of commutators of elements of $\pi_1(\Sigma)$. (Exercise 11(2) is an example of a bracket of the form $[A, \bar{A}]$)*

Jointly with Gadgil we showed that the Goldman bracket, together with the power operation, can count intersections and self-intersections in all orientable surfaces (with or without boundary). The following two theorems are special cases of our results.

THEOREM 5.6 ([**6**]). *If a and b are two distinct free homotopy classes in an oriented surface (with or without boundary), then there exists a constant $c \in \mathbb{R}$ such that the intersection number of a and b is equal to the number of terms (counted with multiplicity) of the bracket $[a, b^q]$ divided by q, for all $q \geq c$. In symbols, $M[a, b^q] = q \cdot \mathrm{i}(a, b)$, for all q, such that $q \geq c$.*

THEOREM 5.7 ([**6**]). *If a is a free homotopy class in an oriented surface (with or without boundary), which is not the power of other class then there exists a constant $c \in \mathbb{R}$ such that the self-intersection number of a is equal to the number of terms (counted with multiplicity) of the bracket $[a, a^q]$, divided by $2 \cdot q$ for all $q \geq c$. In symbols, $M[a, a^q] = 2 \cdot \mathrm{SI}(a)$ for all $q \geq c$.*

EXERCISE 13. *Otal [**16**] showed that given A and B in π_0, there exists $C \in \pi_0$ such that $i(A, C) \neq i(B, C)$. Combine this result with one of the theorems in this section to prove that given two distinct free free homotopy classes of curves A and B, then there exists a third class $C \in \pi_0$ such that $[A, C]$ is not equal to $[B, C]$. (Hint: For any pairs of non-power classes X, $Y \in \pi_0$ and any positive integer q, $\mathrm{i}(X, Y^q) = q \cdot \mathrm{i}(X, Y)$.)*

6. Open problems and generalizations

6.1. The structure of the Goldman Lie algebra. The Goldman bracket determines an infinite-dimensional Lie algebra and not much is known about its structure. This Lie algebra is not, except in the case of the torus, one of the better understood Lie algebras like those of Kac-Moody [**12**].

Regarding the structure of these Lie algebras, Etingof [**7**] using algebraic tools proved the following.

THEOREM 6.1. *The center of the Goldman Lie algebra of a closed oriented surface is the one dimensional subspace generated by the trivial loop $\circ$.*

Kawazumi and Kuno [**13**] studied the center of the Goldman Lie algebra on a surface of infinite genus and one boundary component. This surface is constructed as follows: By identifying the points in the boundary of a compact oriented surface of genus g with one boundary with the points in one of the boundaries of a surface of genus one and two boundary components, one obtains a surface of genus $g+1$ with one boundary component and an embeding $i_g\colon \Sigma_{g,1} \longrightarrow \Sigma_{g++1,1}$. The surface $\Sigma_{\infty,1}$ is the inductive limit of these embeddings.

THEOREM 6.2. *The center of the Goldman Lie algebra of the surface Σ_∞ is the one dimensional space generated by the trivial loop.*

If a surface has non-empty boundary, it is not hard to see that linear combinations of conjugacy classes of curves parallel to the boundary components are in the center. Hence, it seems reasonable to conjecture that the center consists of linear combinations of all conjugacy classes parallel to boundary components (that is, all powers of boundary components). It would be interesting to use the results described above to give a complete characterization of the center of the Goldman Lie algebra. The main obstacle one finds when trying to study this problem geometrically is that linear combinations (with signs) of free homotopy classes of curves are not geometric objects. (Recall that for each curve a, $\mathcal{FH}(\bar{a}) \neq -\mathcal{FH}(a)$).

OPEN PROBLEM 1. *Characterize the center of the Goldman Lie algebra on surfaces with boundary of finite type.*

OPEN PROBLEM 2. *An element $A \in \pi_0$ determines a linear map* ad_A *from* $\mathbb{Z}[\pi_0]$ *to itself, defined by* $\mathrm{ad}_A(B) = [A, B]$. *Characterize the kernel of* ad_A *for each* $A \in \pi_0$.

OPEN PROBLEM 3. *Decide whether the Goldman Lie algebra is finitely generated.*

Along with the Goldman Lie algebra of loops on a surface Σ, the *homological Goldman Lie algebra on* Σ introduced by Goldman [**10**] is defined as follows: Denote by H, the first homology group of Σ with coefficients in $\mathbb{Z}$ and by $\mathbb{Z}[H]$ the free $\mathbb{Z}$-module on H. The bracket of two elements α, β in the basis H is defined as $(\alpha, \beta) \cdot \widehat{\alpha+\beta}$,where (α, β) denotes the intersection pairing and $\widehat{\alpha+\beta}$ the homology class of $\alpha+\beta$ in H. There is a natural map from $\mathbb{Z}[\pi_0]$ to $\mathbb{Z}[H]$.

Toda [**19**] characterized the ideals of the homological Goldman Lie algebra $\mathbb{Z}[H]$ tensored with $\mathbb{Q}$. Kawazami, Kuno and Toda [**15**] proved that $\mathbb{Z}[H]$ tensored with $\mathbb{Q}$ is finitely generated if the intersection pairing is nondegenerate.

6.2. The Goldman Lie algebra and the intersection of curves on surfaces. By computer experiments, (using the presentation of the Goldman Lie algebra in [**2**]) we showed that in a few surfaces with boundary and fundamental group with relatively small number of generators (at most 4), if A is represented by a relatively short word (of at most 16 letters) which is not a proper power then $M[x, \bar{x}] = 2 \cdot \mathrm{SI}(x)$ (see Example (b)). This lead us to the following:

OPEN PROBLEM 4. *Prove the following conjecture: If $A \in \pi_0$ is not a power of another class then $M[A, \bar{A}] = 2 \cdot \mathrm{SI}(A)$. Namely, twice the self-intersection number is the number of terms of the bracket of a class of a curve with its inverse.*

Open Problem 5. *By Exercise 12, for each $A \in \pi_0$, the terms of the bracket $[A, \bar{A}]$ are commutators. It would be interesting to understand better this phenomenon, and how it relates to the lower central series of the fundamental group.*

One knows that up to deformation, (see, for instance [**8**]) that any self-homeomorphism of a surface is a composition of so called Dehn twists along embedded curves. (Recall that a Dehn twist about an embedded curve a is obtained by cutting along a, twisting $360°$ and reglueing.) In [**14**] a formula for the action of a Dehn twist on the group ring of the fundamental group of a surface was found (after a certain completion). The formula uses a string topology operation like that used in the Goldman bracket [**17**].

It is interesting that the formula makes sense for non-embedded curves, defining "generalized Dehn twists in the (completed) group ring", [**14**].

6.3. The Goldman Lie algebra and the mapping class group. In another direction, Gadgil [**9**] proved the following result:

Theorem 6.3. *A homotopy equivalence $f\colon \Sigma \longrightarrow \Sigma'$ between two compact, connected, oriented surfaces Σ, Σ' is homotopic to a homeomorphism if and only if it commutes with the Goldman bracket. In symbols, for all α, β in $\mathbb{Z}[\pi_0(\Sigma)]$,*

$$[f^*(x), f^*(y)] = f^*([x, y]),$$

where $f^*\colon \mathbb{Z}[\pi_0(\Sigma)] \longrightarrow \mathbb{Z}[\pi_0(\Sigma')]$ is the map induced by f.
The interesting case is when the surface has boundary.

Also, in relation with the mapping class group, in [**3**] we prove the following:

Theorem 6.4. *Let Ω be a bijection on the set π_0 of free homotopy classes of closed curves on an oriented surface. Suppose the following*

(1) Ω preserves simple curves.
(2) If Ω is extended linearly to the free $\mathbb{Z}$-module generated by π_0, $\mathbb{Z}[\pi_0]$ then Ω preserves the Goldman Lie bracket. In symbols $[\Omega(x), \Omega(y)] = \Omega([x, y])$ for all x, y in π_0.
(3) For all all x, in π_0, $\Omega(\bar{x}) = \bar{\Omega(x)}$. ($\bar{x}$ denotes the inverse of x.)

Then the restriction of Ω to the subset of simple closed curves is induced by an element of the mapping class group. Moreover, if the surface is not $\Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}$ or $\Sigma_{0,4}$ then the restriction of Ω to the subset of simple closed curves is induced by a unique element of the mapping class group.

6.4. String topology. The ideas of Goldman [**10**] and Turaev [**20**] prompted the author in collaboration with Dennis Sullivan to notice these structures exist for all oriented manifolds M. The underlying modules are various homology groups (ordinary and equivariant) of the space of all smooth mappings of the circle into M, or certain natural subsets thereof [**4**], [**1**]. There extensions lead to algebraic activity related to algebras with duality.

Remark 6.5. Since there is a natural bijection between the set of free homotopy classes of curves on a surface and the components of the free loop space, we haved denoted the set of free homotopy classes by π_0 . Linear combination of these components form the zeroeth ordinary homology and equivariant homology of the free loop space.

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