Topic Title

Homework: Compute the Gaussian curvature at a point in the sphere of radius R. If you feel brave, compute also the curvature at a point in the hyperbolic plane (using one of the models).

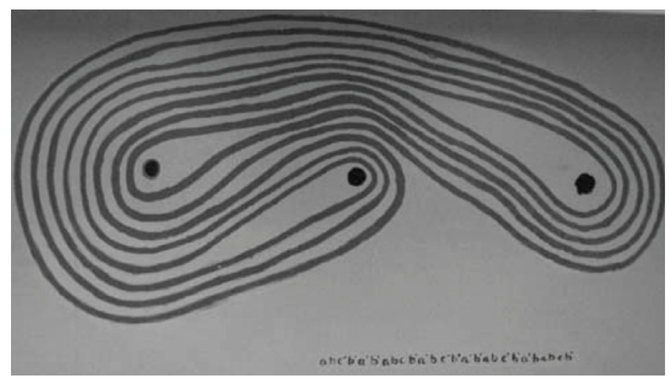
$$K(p) := 3\left(\lim_{r\to 0+} \frac{2\pi r - C(r,p)}{\pi r^3}\right)$$

C(r,p) is the length of the circle of radius r and center p.
K(p) is the Gaussian curvature at p

Many people have an impression that mathematics is an austere and formal subject concerned with complicated and ultimately confusing rules for the manipulation of numbers, symbols, and equations, rather like the preparation of a complicated income tax return. **Good mathematics is quite opposite to this. Mathematics is an art of human understanding.** ... Our brains are complicated devices, with many specialized modules working behind the scenes to give us an integrated understanding of the world. **Mathematical concepts are abstract, so it ends up that there are many different ways they can sit in our brains. A given mathematical concept might be primarily a symbolic equation, a picture, a rhythmic pattern, a short movie — or best of all, an integrated combination of several different representations.**

Bill Thurston

Simple curves can be complicated

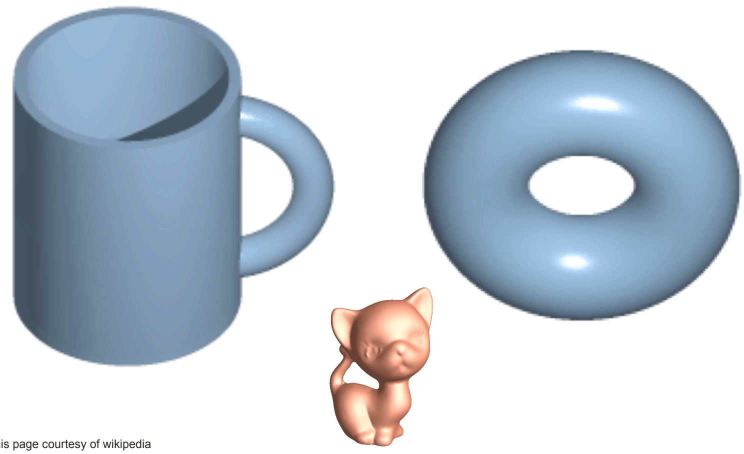


Summary of our first discussion

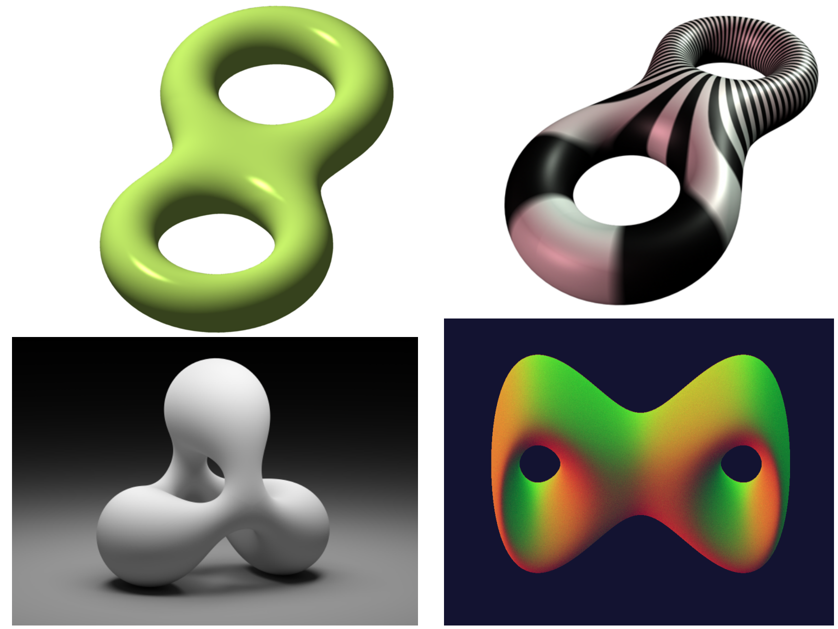
- Fix a hyperbolic surface S.
- For each $k \geq 0$, define $l_k(S)$ as the length of the shortest closed geodesic in S with at least k self-intersection points.
- Prove that $l_k(S)$ is increasing.

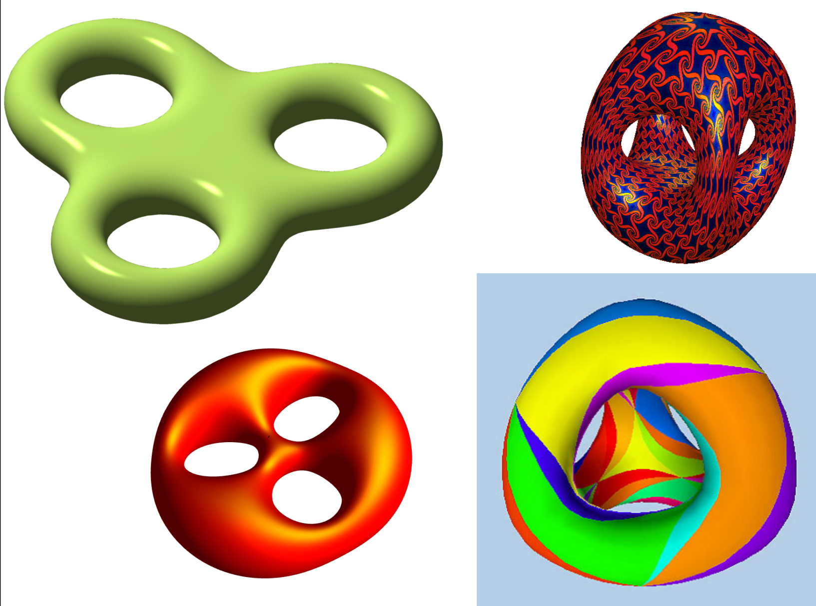
Surfaces

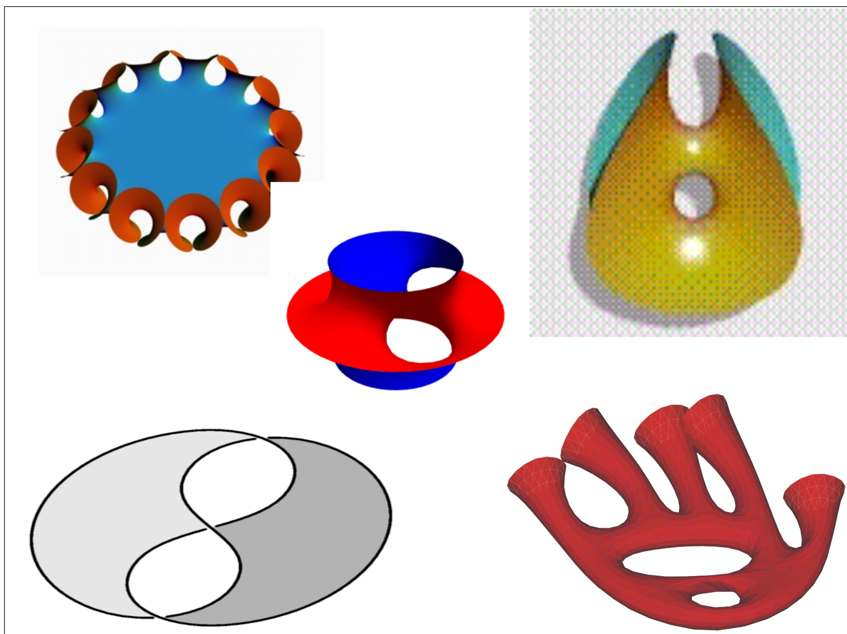
Topology jest: A topologist can't distinguish a coffee mug from a doughnut

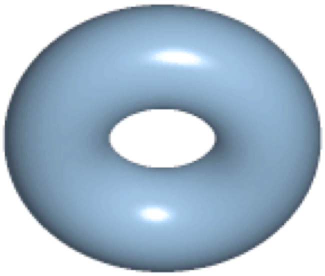


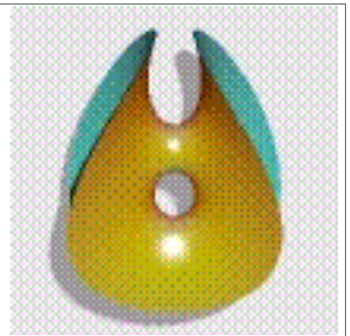
Animation on this page courtesy of wikipedia

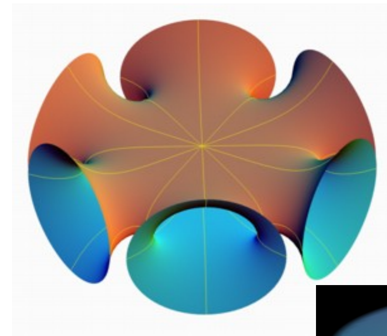


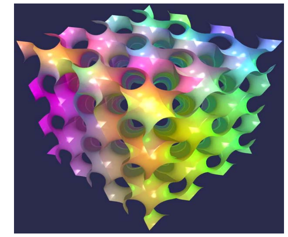


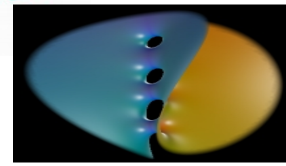


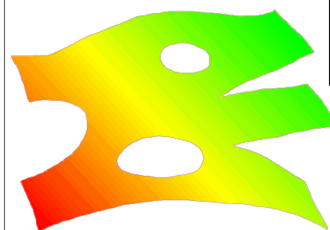


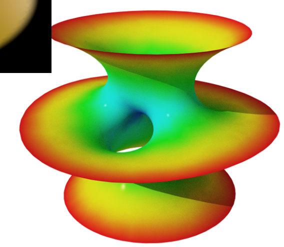


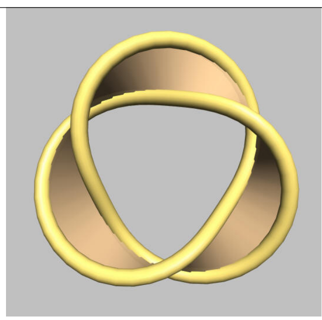


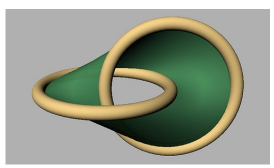


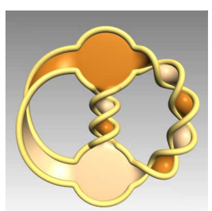




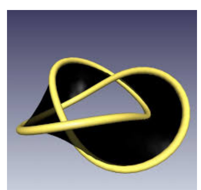


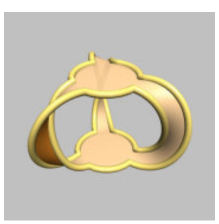


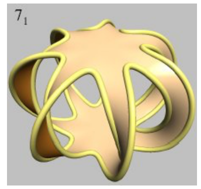




"(1,3,5) Pretzel Knot," by Jarke J. van Wijk (Technische Universiteit Eindhoven). Image courtesy of Jarke J. van Wijk.







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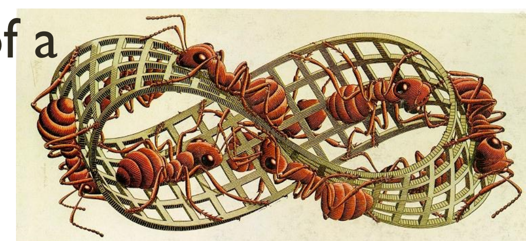
Orientation of a surface



Möbius Strip II
1963 Woodcut
M.C. Escher

A surface is orientable if it is "two sided" or equivalently, if it does not contain a Möbius strip.

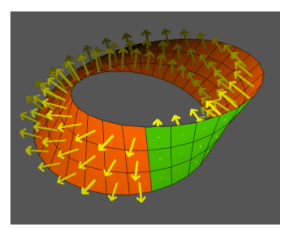
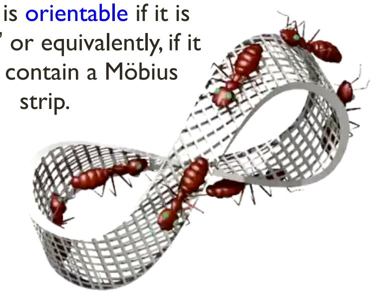
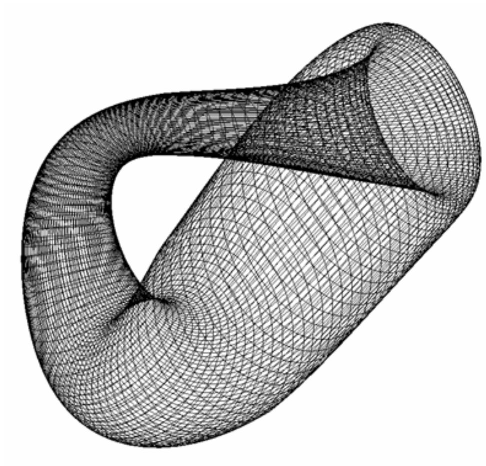


Image Konrad Polthier)

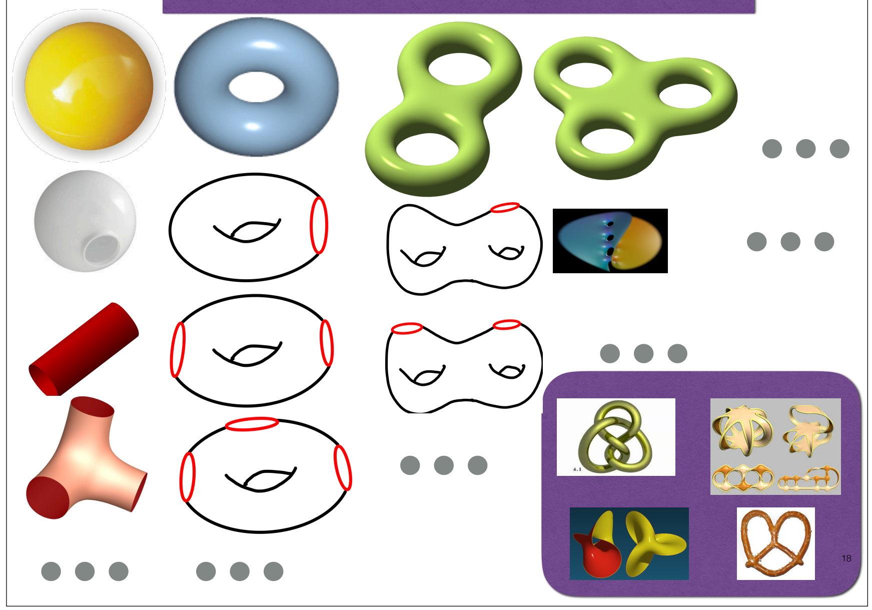


The Klein bottle, our "last" non-orientable surface



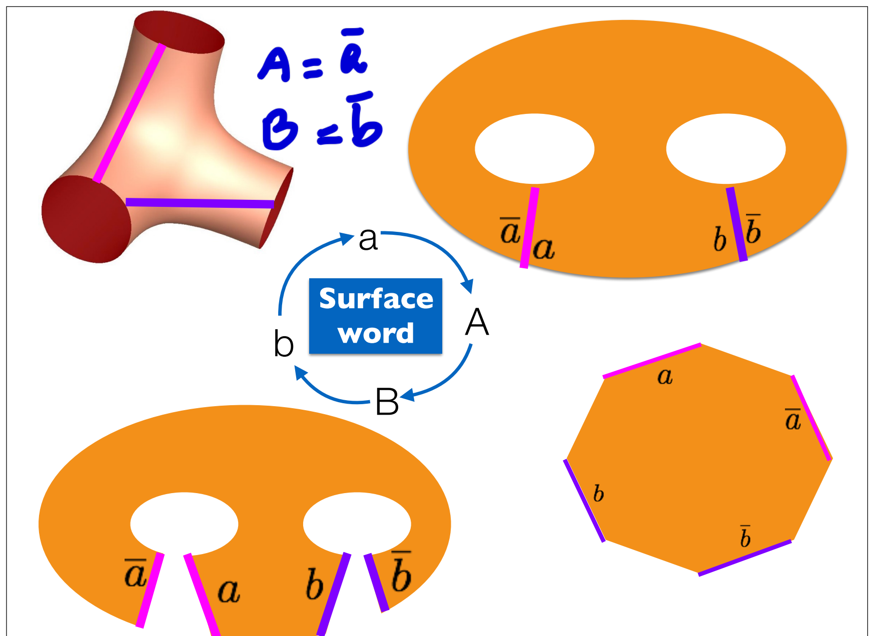
17

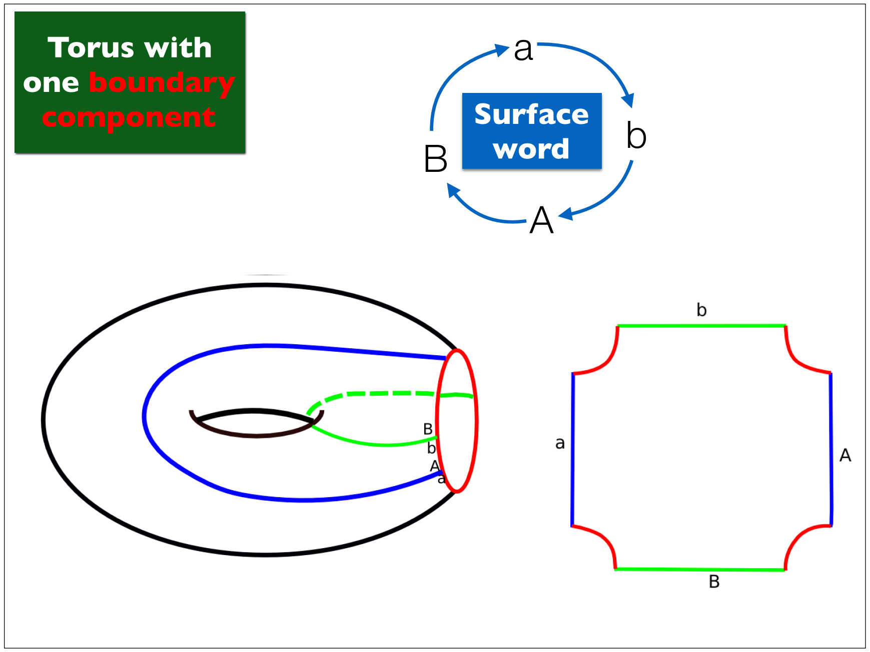
Classification of surfaces

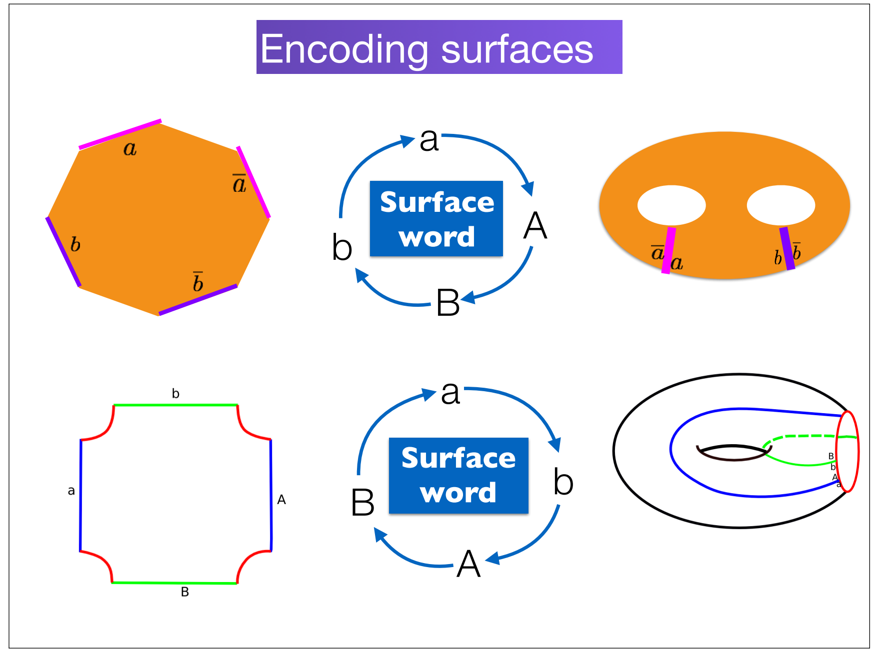


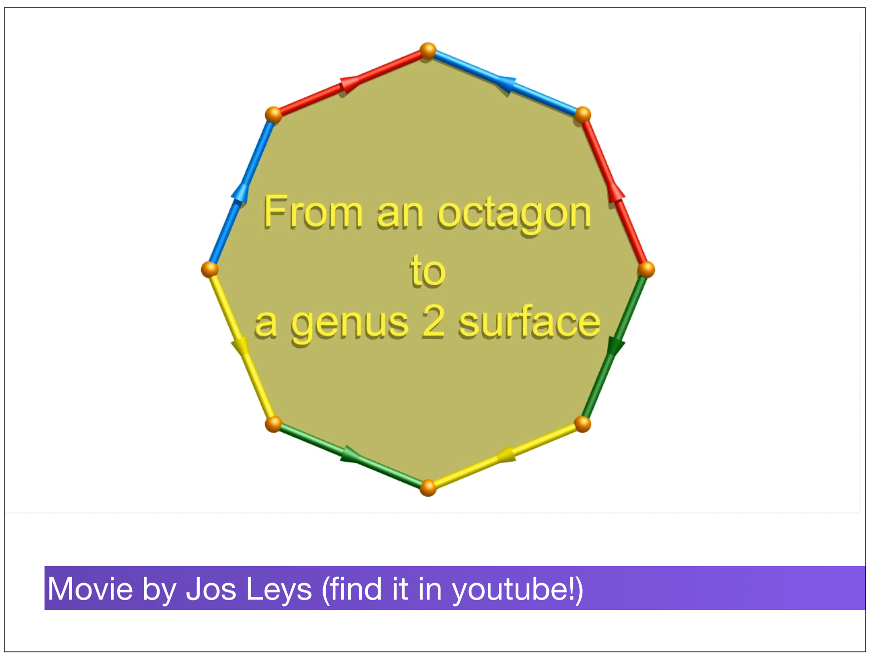
18

Encoding surfaces





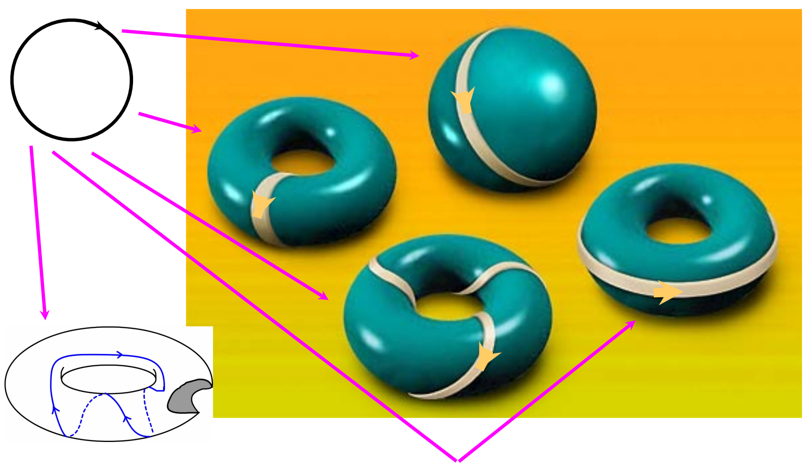




Curves on surfaces

Curves on a surface

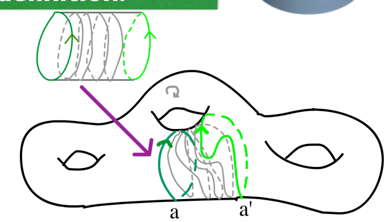
You can think of a closed curve as a loop drawn on a surface



Deforming (homotoping) a curve on a surface

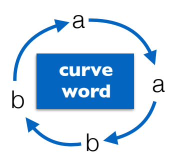
Homework: Explain formal definition of homotopy in terms of the the "movie" definition.

We want to study closed curves on a given surface up to deformation

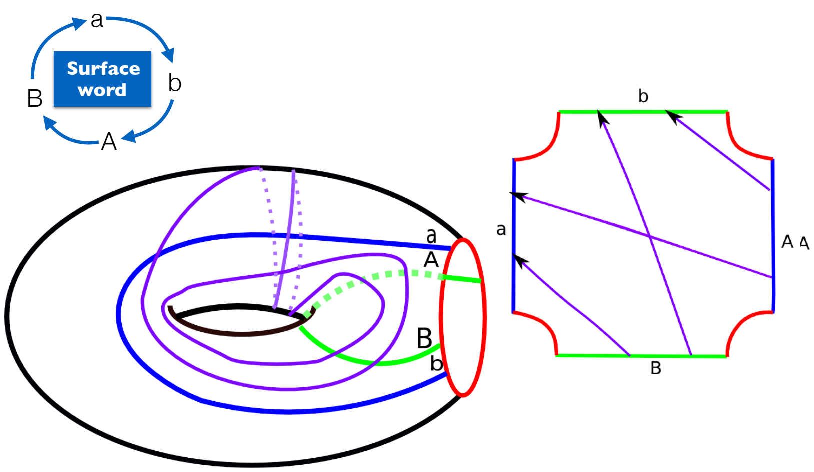


a is freely homotopic to a'

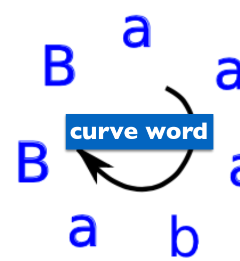
Torus with one boundary component



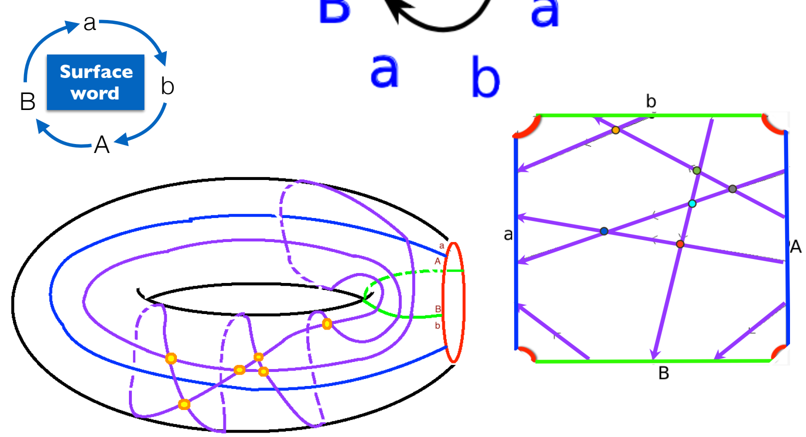
aabb

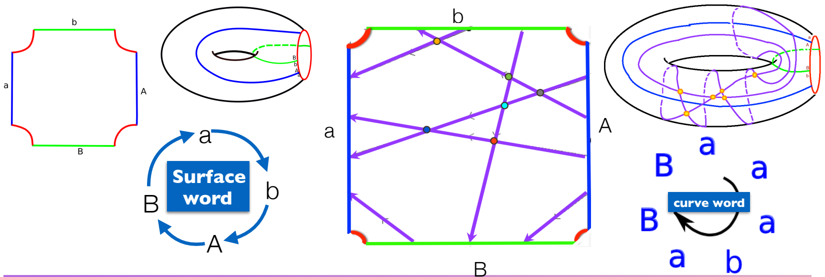


Torus with one boundary component



aaabaBB





A ***surface word*** is a ring of letters. Every letter appears exactly once in lower case, and exactly once in upper case.

Fix a surface word W. A ***curve word*** on S is a *reduced* ring of letters word on the letters of W.

deformation classes of closed oriented curves on a surface with boundary S

one to one

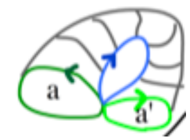
Curve words in the alphabet of S

π_0 (π_0=(S))denotes the set of deformation classes of curves on a surface S.

π_0 = set of deformation classes of closed oriented curves on a surface S

Bijection

conjugacy class of π_1 (S)



Bijection when the surface S has boundary

Curve words in the surface alphabet (cyclic, reduced)

Counting

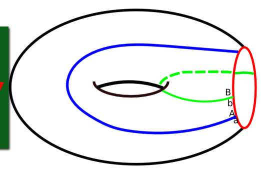
Fix a surface word for a surface S. The ***word length of a deformation class of curves on S*** is the number of letters on the curve word.

How many deformation classes of curves of word length L on the torus with one boundary component?
And on the pair of pants?

Counting all deformation classes of curves

WL	sum of all words	sum of non-power words	3^WL/WL
1	4	4	3.00
2	8	4	4.50
3	12	8	9.00
4	26	18	20.25
5	52	48	48.60
6	132	116	121.50
7	316	312	312.43
8	836	810	820.13

Torus with one boundary component

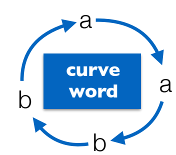


Crossings of curves on surfaces

Fix a surface S. Consider a deformation class of curves w on S. The *self-intersection number of w* is the smallest number of crossings of representatives of w with double intersections

What is the self-intersection number of the class of the red curve?

Torus with one boundary component



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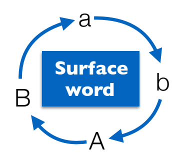
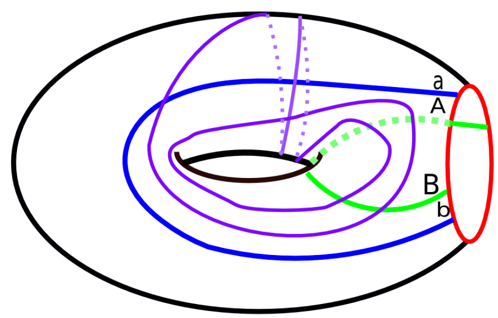
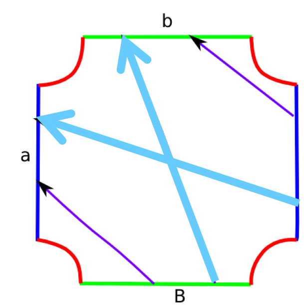
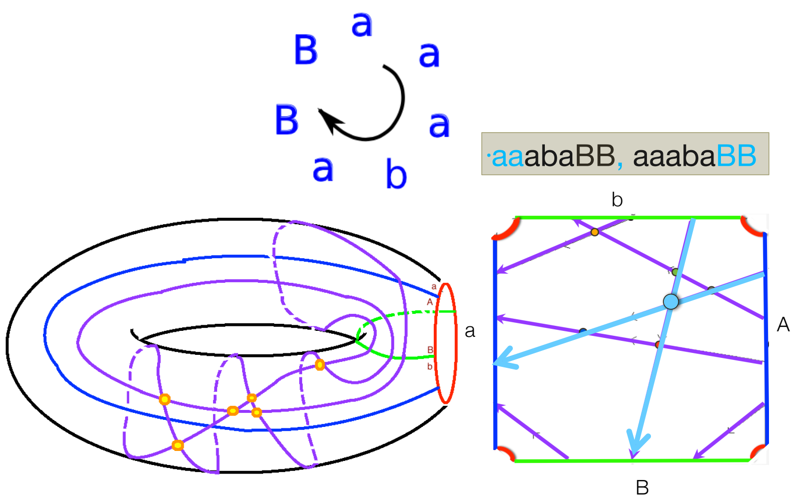
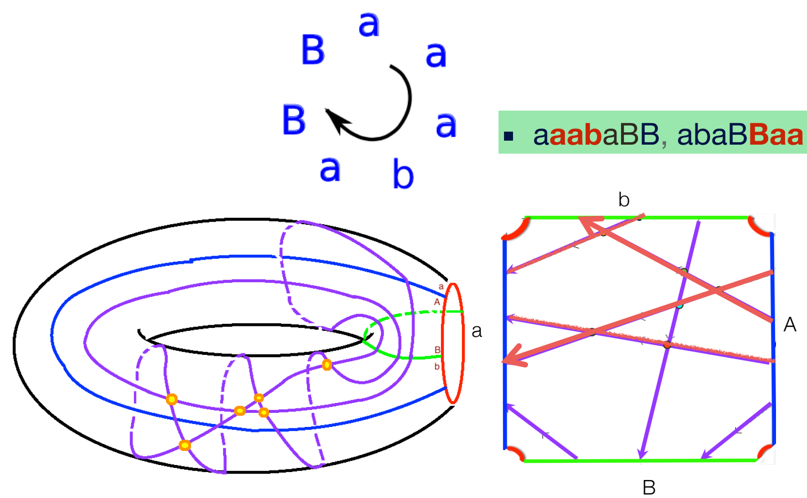
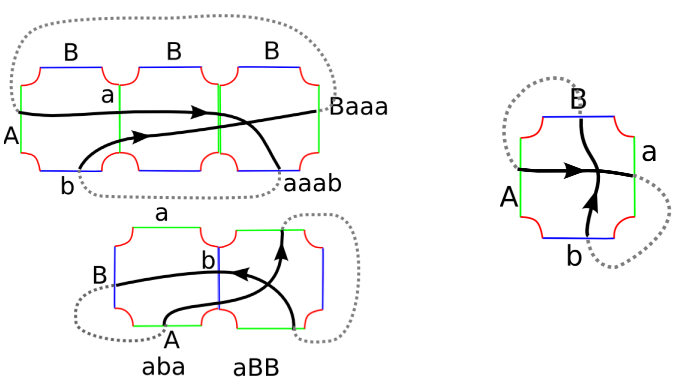








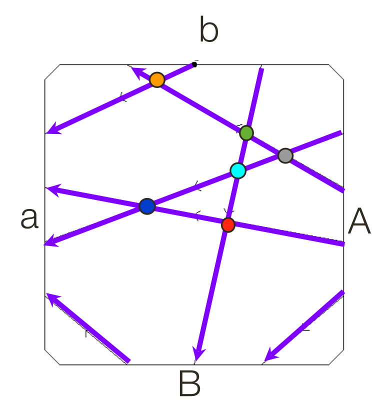




The three kinds of linked pairs on the torus with one boundary component

Linked pairs

- aaabaBB, aaabaBB
- aaabaBB ,aaabaBB
- aaabaBB, abaBBaa
- BaaabaB,aaabaBB
- aaabaBB , aabaBBa
- aaabaBB,aaabaBB

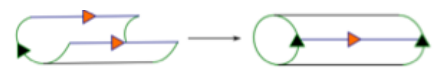


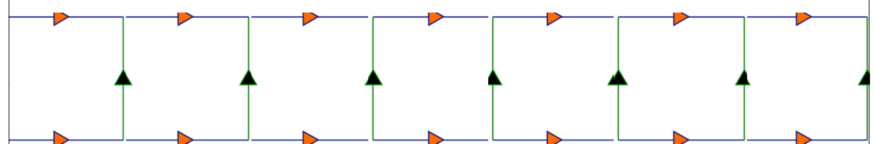
Theorem (Birman-Series, Cohen-Lustig, C. α representative of deformation class v with minimal self-intersection number,

Self-intersection points of α ⟷ one-to-one correspondence ⟷ linked pairs of v

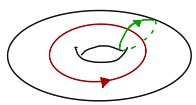
(Hyperbolic) Metrics on surfaces

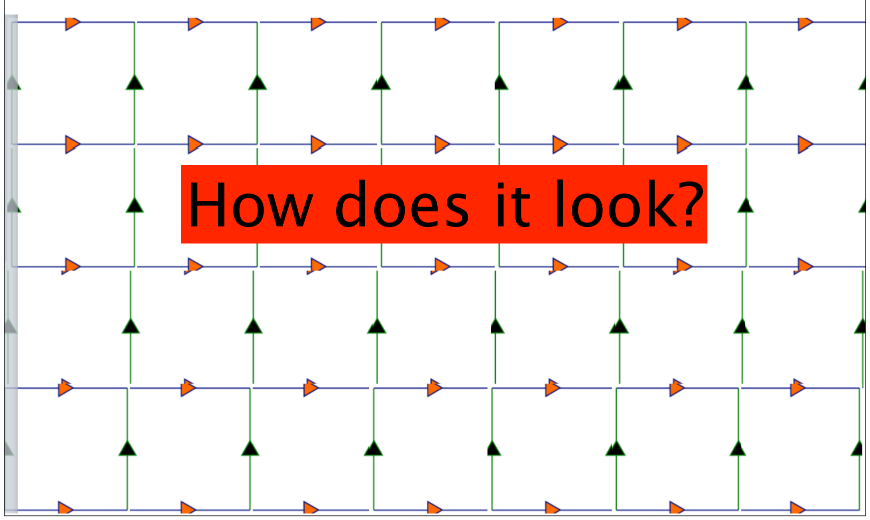
Parenthesis: What happens with curves on a cylinder when you “unroll” it?





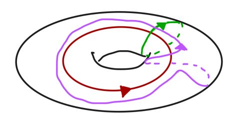
The torus with pac-person metric

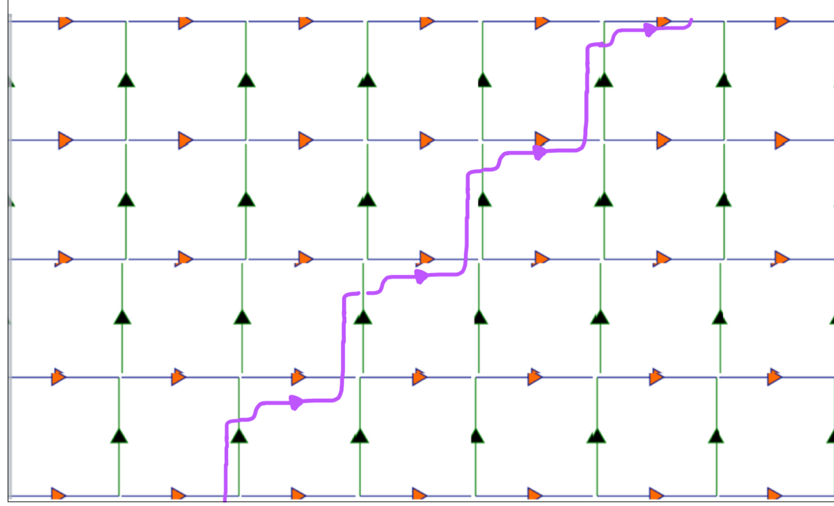




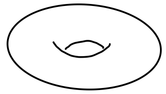
How does it look?

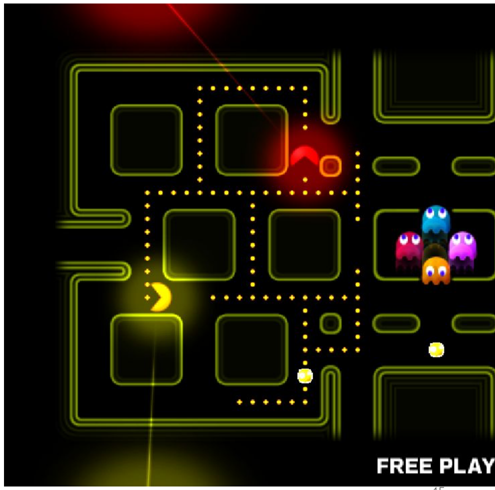
Parenthesis’: What happens with curves on a torus when you “unroll” it?



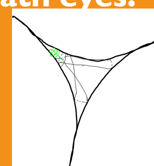


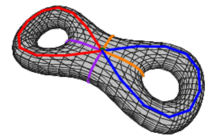
Pac-person metric on the torus





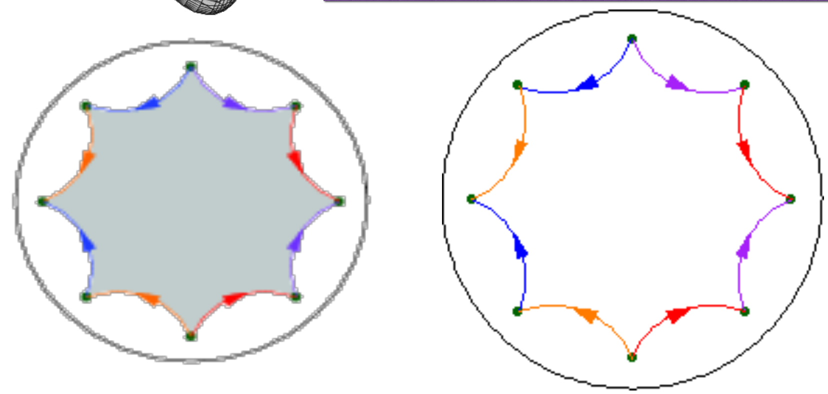
Analgously, there is a metric on any hyperbolic surface in which the landscape at each point looks “the same” for our math eyes.





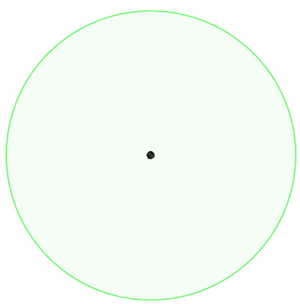
“Unrolling” a closed surface.

We can endow the genus 2 surface with pac-person metric!



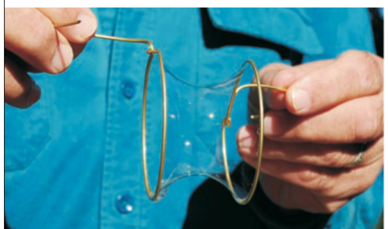
Rowland, Todd. "Universal Cover." From MathWorld--A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/UniversalCover.html

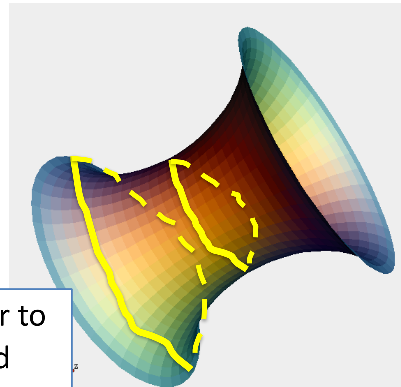
A regular octogon in the Poincarè disk with interior angle 135°



Length of curves on surfaces

In surfaces of constant negative curvature (that is, hyperbolic), every deformation class of curves contains a unique shortest curve. This shortest curve is called geodesic.





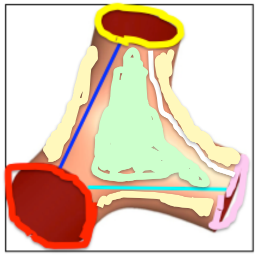
One can associate a real number to each deformation class of closed curves: the length of the shortest curve in the class.

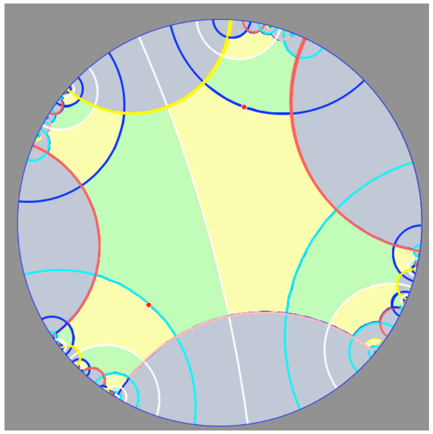
More about length, later in this presentation

Determine (or find bounds of) the Hausdorff dimension of the limit set of a hyperbolic pair of pants in terms of the length of the boundary components.

hyperbolic pair of pants in terms of the length of the boundary components.

Unrolling the pair of pants.
Endow it with the pac-person metric





G=< R_1R_2, R_1R_3>= $\left\langle \begin{pmatrix} \frac{1+a}{1-a} & \frac{2a}{1-a} \\ \frac{2}{1-a} & \frac{1+a}{1-a} \end{pmatrix}, \begin{pmatrix} \frac{b+c}{-b+c} & -\frac{2bc}{-b+c} \\ -\frac{2}{-b+c} & \frac{b+c}{-b+c} \end{pmatrix} \right\rangle$

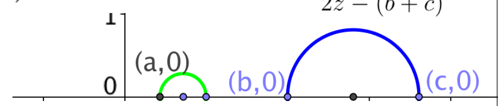
Pants ~ H^2/G

MASKIT'S PARAMETRIZATION OF THE PAIR OF PANTS.

A=(0,a),
B=(b,0),
C=(c,0)
a<1<b<c

$$R_1(z) = -\bar{z}.$$

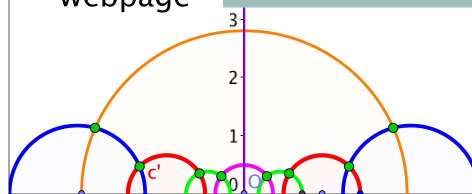
$$R_3(z) = \frac{(b+c)\bar{z} - 2bc}{2\bar{z} - (b+c)}.$$

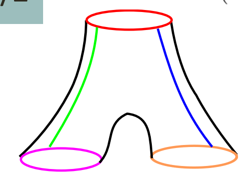


Reference in webpage

Find lengths of curves using cosh[L/2]=trace[L]/2

$$R_2(z) = \frac{(1+a)\bar{z} - 2a}{2\bar{z} - (1+a)}.$$



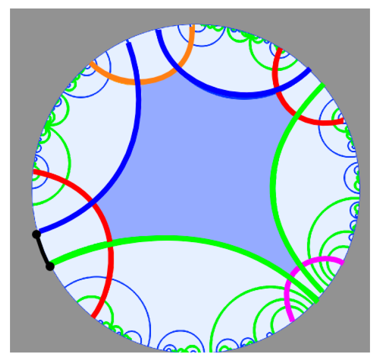


Find lengths of curves using cosh[L/2]=trace[L]/2

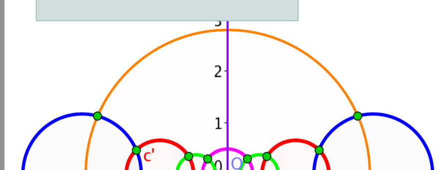
Recipe (to understand later):

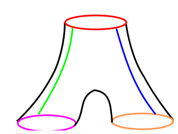
1. Associate to each curve a curve word.
2. Find the appropriate matrix associated to each letter of your curve word.
3. Replace each letter of your curve word with the matrices you found in 2.
4. Use the formula above to obtain L, the length of your curve.

Margulis, Huber, Delsarte, Selbert, Lalley…: #{w in π_0, gl(w) ≤ L} ~ e^{hL}/hL



h is the Hausdorff dimension of the limit set



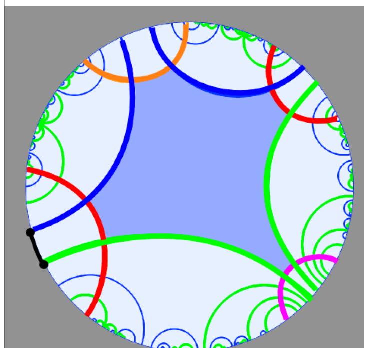


Counting curves

Margulis, Huber, Delsarte, Selbert, Lalley…: $\#\{w \text{ in } \pi_0, gl(w) \leq L\} \sim e^{hL}/hL$

Find a "something like" formula (or similar) for h.

h is the Hausdorff dimension of the limit set



la	lb	lc	Total Curves	WL	C	MAX HL	h
20	20	30	14	5	15	75	0.05378
20	20	30	408	10	15	150	0.05402
20	20	30	32258	15	15	225	0.05752
2	2.5	3	6	5	1.5	7.5	0.03778
2	2.5	3	144	10	1.5	15.0	0.04601
2	2.5	3	3055	15	1.5	22.5	0.04602
1	4	8	2	5	1	5	-
1	4	8	5	10	1	10	0.01695
1	4	8	18	15	1	15	0.01939

We will use Box Counting Dimension because in our example both (H and BC coincide) and BC is easier to swallow

~~Hausdorff dimension~~

Box Counting Dimension

Suppose that $N(\varepsilon)$ is the number of boxes of side length ε required to cover the set. Then the box-counting dimension is defined as:

$$\dim_{\text{box}}(S) := \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}.$$

Determine (or find bounds of) the Hausdorff dimension of the **limit set** of a hyperbolic pair of pants in terms of the length of the boundary components.

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