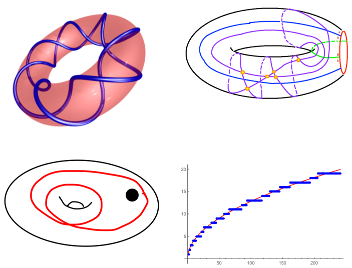


Hyperbolic Geometry

Moira Chas

A Combinatorial Model of something

In short, mathematics only exists in a living community of mathematicians that spreads understanding and breaths life into ideas both old and new.

The question of who is the first person to ever set foot on some square meter of land is really secondary.

Revolutionary change does matter, but revolutions are few, and they are not self-sustaining --- they depend very heavily on the community of mathematicians.

Bill Thurston

When you understand something really well, if that something is unknown is called research, if it is know, it is called learning.

Dennis Sullivan

~~What is geometry?~~

What do I mean by geometry?

What do ***you*** mean by geometry?

Properties of straight lines

- "No angle"
- locally minimizes distances
- direction does not change
- the landscape does not change as we walk along.

Can yo find curves with the same properties on a sphere?

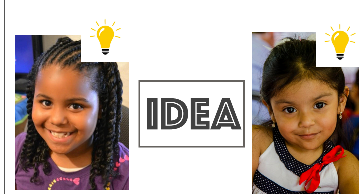
What is "a circle"?

How can you check a curve is a circle?

What is "a straight line"?

How can you check a curve is a straight line?

Suppose you are a tiny bug walking along a curve. No matter how much you move (forwards or backwards) the curve looks the same curve looks the same. What is the curve?



Idea (we learned from Thurston): When you are trying to understand a space, put yourself as a point in this space

You are on a circle, made of a magic stretching rubber band material.
Describe how the circle looks like as the center moves (on a straight line) further and further away from you?

Idea (we learned from Thurston): When you are trying to understand a space, put yourself as a point in this space

You are on a circle on a sphere (of radius 1). The circle is made of a magic stretching rubber band material (and does not leave the sphere).
Describe the circle as the center moves further and further away from you?

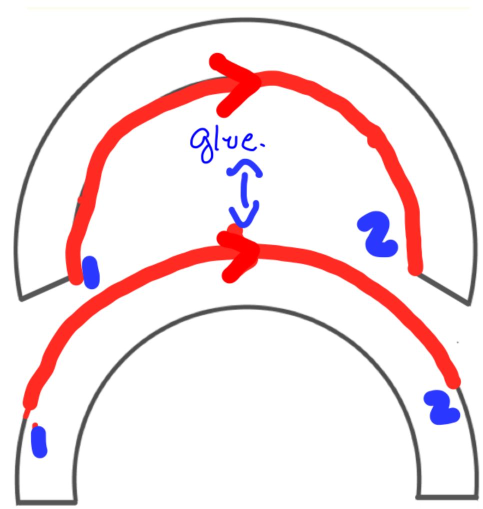
Idea (we learned from Thurston): When you are trying to understand a space, put yourself as a point in this space

What are all the curves of constant curvature through you, when you are on

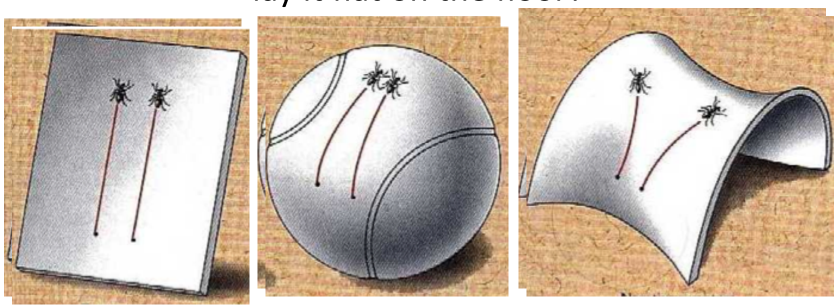
1. a plane?
2. a circle?

(with a certain tangent at you?

Idea (we learned from Thurston): When you are trying to understand a space, put yourself as a point in this space



Can you cut the surface in such a way that you can lay it flat on the floor?

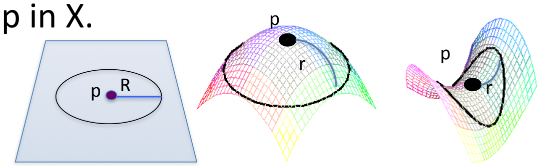


There are three kinds of geometry which possess a notion of distance and which look the same from any viewpoint with your head turned in any orientation. W. Thurston

Curvature in a two dimensional space (where we can measure length of curves)

- Consider a two dimensional space X and a point p in X.
- Let C(p,r) be the length of a circle of radius r centered at p in X.

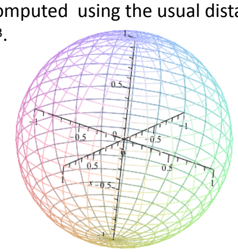
Longer, shorter or equal to $2\pi r$?

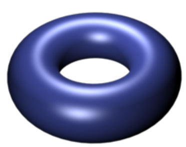


Define the ***curvature K(p)*** at p, as

$$K(p) := 3\left(\lim_{r\to 0+} \frac{2\pi r - C(r,p)}{\pi r^3}\right)$$

The sphere is in R^3. Thus, using coordinates, and a bit of calculus the distance between points can be computed using the usual distance in R^3.

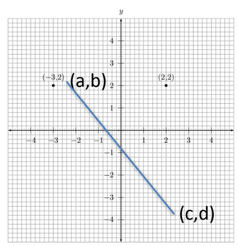




The torus is in R^3. As in the sphere the distance between points can be computed using the usual distance in R^3.

Note: not every point on the torus "looks" the same.

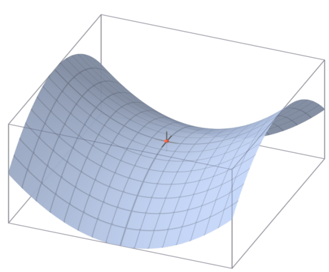
In the plane, distance between points (a,b) and (c,d) is $((a-c)^2+(b-d)^2)^{1/2}$



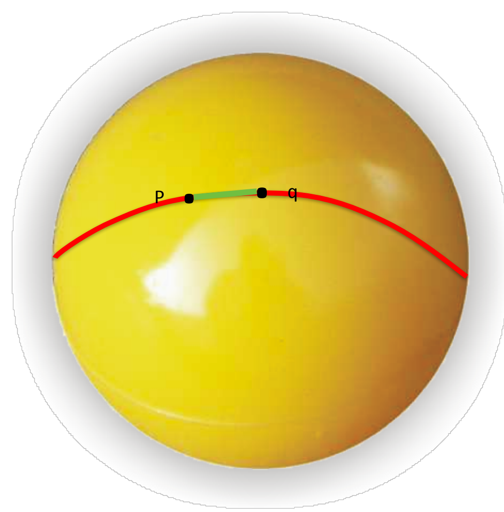
We can "put" (isometrically embed) a bounded piece of "saddle plane" in R^3 but we cannot not put (isometrically embed) it an unbounded saddle space without "crumpling" it.

The hyperbolic plane is an unbounded saddle plane, where every point looks the same in all directions.

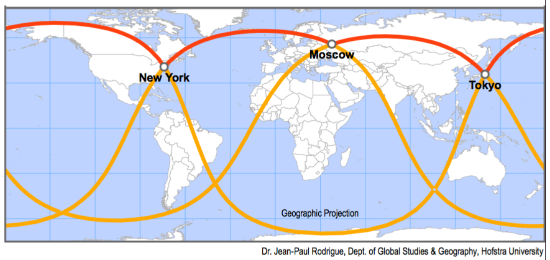
We can use **models** (maps), in which "what you see is what you get" is not true.

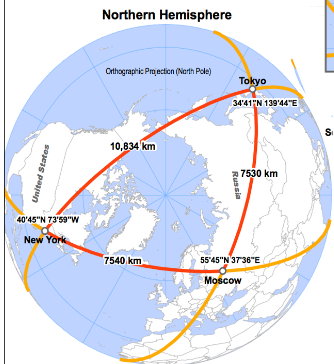


What is the shortest path from the point p to the point q? (p and q are not far from each other)



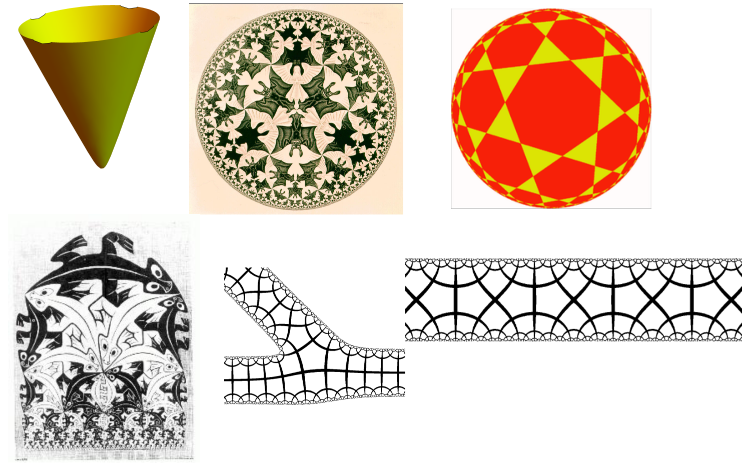
In this sense, this path is straight. It is part of a ***geodesic***.





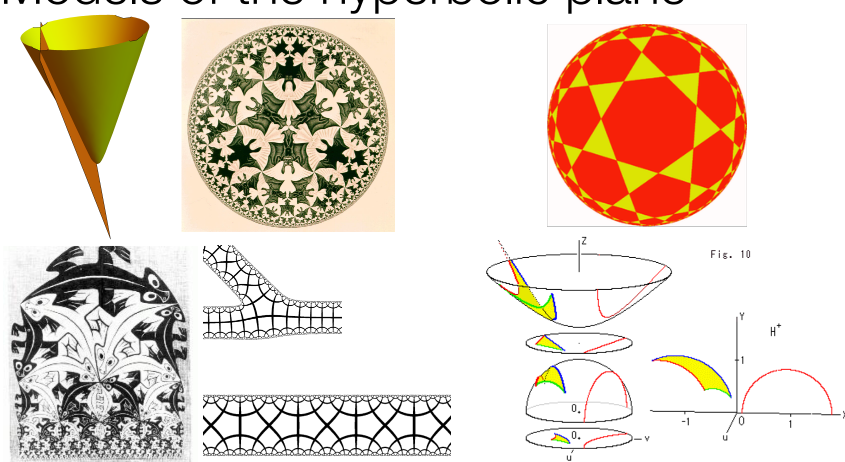
“Straight” lines
(geodesics) in the sphere

Some Models of the hyperbolic plane



Images by Escher and Bulatov

Geodesics(“straigth lines”) in different Models of the hyperbolic plane



Images by Escher and Bulatov

Some models of the hyperbolic plane

H, the Half-space model.

I, the Interior of the disk model.

J, the Jemisphere model (pronounce the J as in Spanish).

K, the Klein model.

L, the 'Loid model (short for hyperboloid).

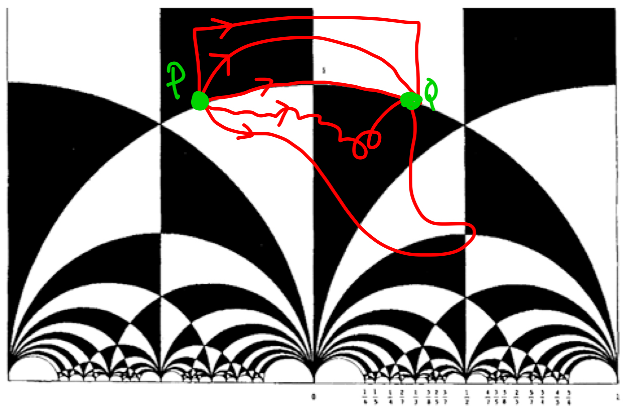
From
http://library.msri.org/books/Book31/files/cannon.pdf

$\gamma : [0,1] \longrightarrow H^2, \gamma(t) = (\gamma_1(t), \gamma_2(t))$

$$\text{length}(\gamma) = \int_0^1 \frac{\sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}dt}{\gamma_2(t)}$$

dist(P,Q)= min{length(γ), γ path from P to Q}

Recall that geodesics are curves locally realizing shortest distance

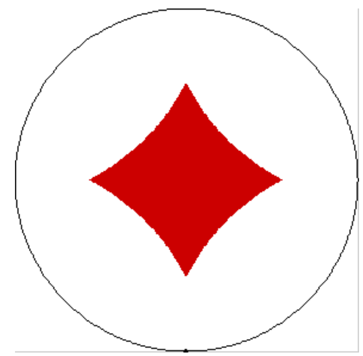


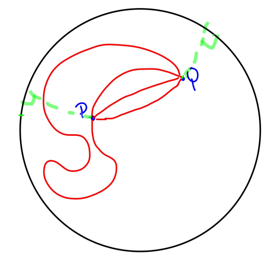
D is the open unit disk in R^2.
For each path

$$\gamma : [0,1] \longrightarrow D, \gamma(t) = (\gamma_1(t), \gamma_2(t))$$

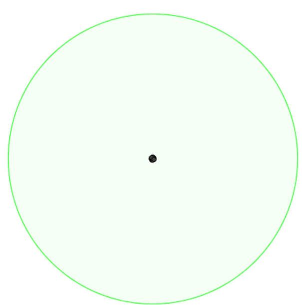
$$\text{length}(\gamma) = \int_0^1 \frac{2\sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}dt}{1 - \gamma_1(t)^2 - \gamma_2(t)^2}$$

The distance between two points p and q in D is the infimum of l(A) over all paths A from p to q .



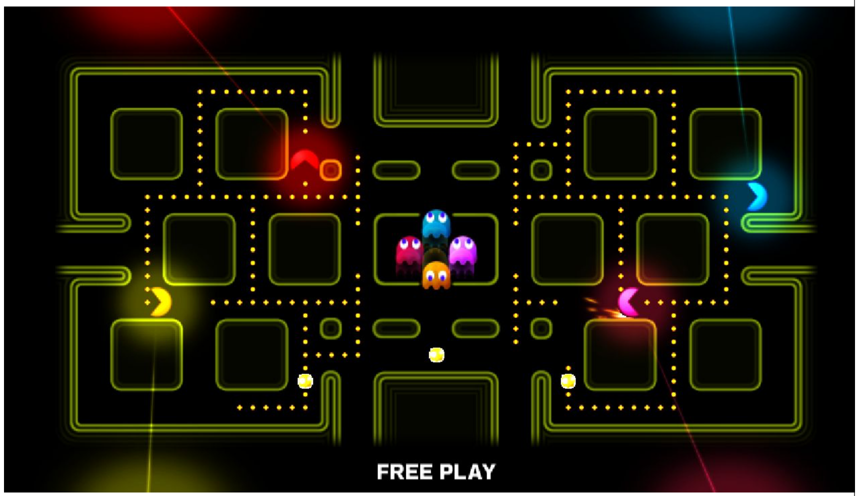


A regular octogon in the
Poincarè disk with interior angle 135°

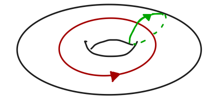


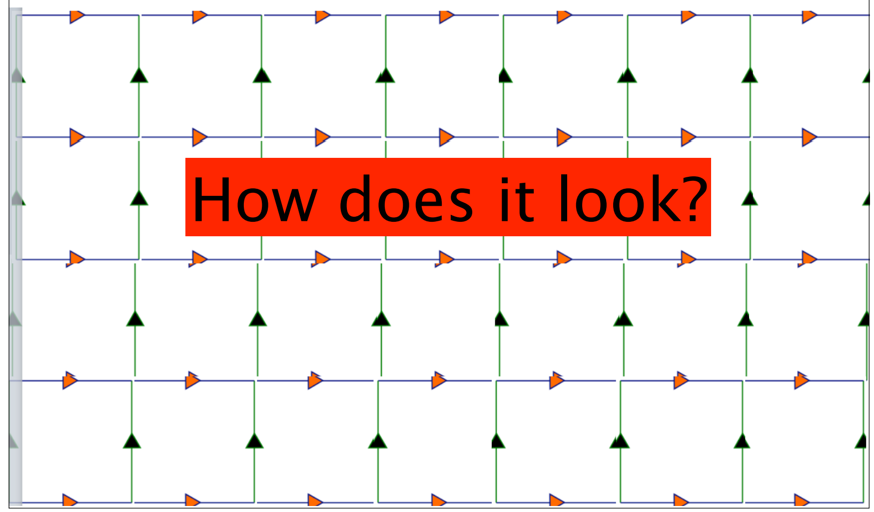
The end

Pac-man metric

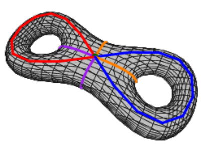


The torus with pac-person metric

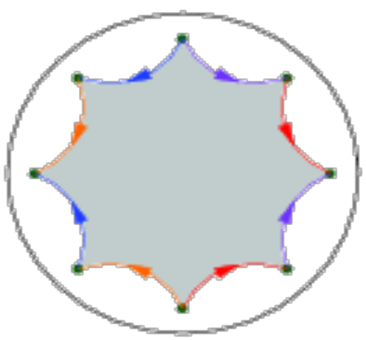


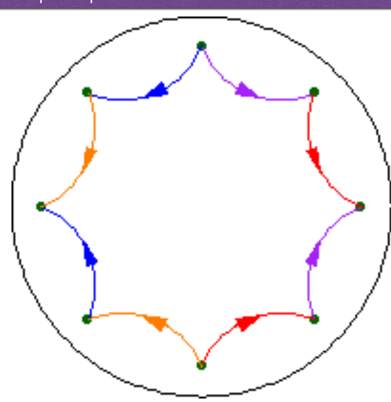


"Unrolling" a closed surface.

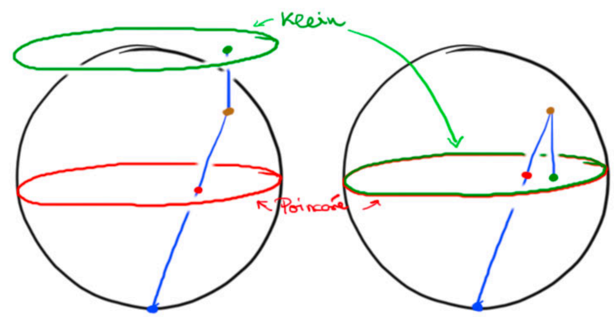


We can endow the genus 2 surface with pac-person metric!

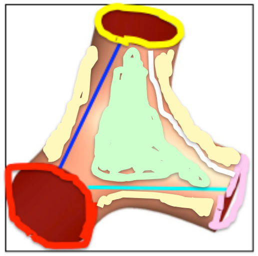


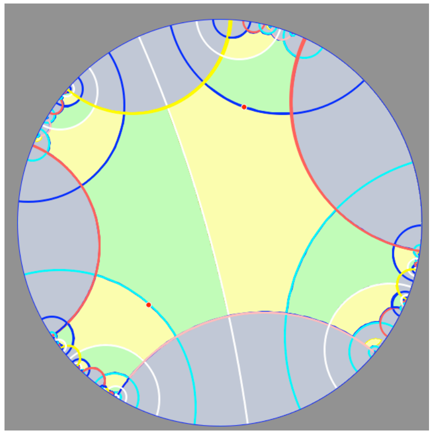


Rowland, Todd. "Universal Cover." From MathWorld--A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/UniversalCover.html

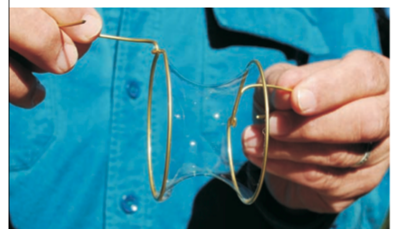


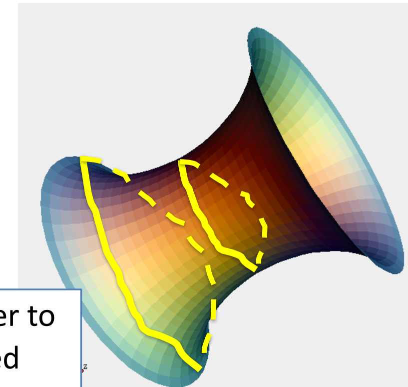
Unrolling the pair of pants.
Endow it with the pac-person metric





In surfaces of constant negative curvature, every deformation class of curves contains a unique shortest curve. This shortest curve is called geodesic.





One can associate a real number to each deformation class of closed curves: the length of the shortest curve in the class.

A hyperbolic metric on the pair of pants is determined by the length of the three (geodesic) boundary components

Pair of pants with geodesic boundary components of different lengths