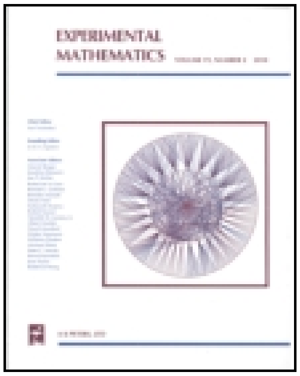

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Experiments Suggesting That the Distribution of the Hyperbolic Length of Closed Geodesics Sampling by Word Length Is Gaussian

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CONTENTS

Each free homotopy class of directed closed curves on a surface with boundary can be described by a cyclic reduced word in the generators of the fundamental group and their inverses. The word length is the number of letters of the cyclic word.

If the surface has a hyperbolic metric with geodesic boundary, the geometric length of the class is the length of the unique geodesic in that class.

By computer experiments, we investigate the distribution of the geometric length among all classes with a given word length on the pair of pants surface. Our experiments strongly suggest that the distribution is normal.

1. INTRODUCTION AND STATEMENT OF THE CONJECTURE

The fundamental group of a pair of pants (that is, a sphere from which three disjoint open disks have been removed) is free on two generators. Fixing a pair of generators a and b of this group, each of which is represented by a simple closed curve parallel to one of the boundary components, then every free homotopy class of closed curves on the pair of pants can be represented uniquely as a reduced cyclic word in the symbols $a, b, \overline{a}, \overline{b}$. (A *cyclic word* w is an equivalence class of words up to a cyclic permutation of their letters; *reduced* means that the cyclic word contains no juxtapositions of a with $\overline{a}$, or b with $\overline{b}$.) The *word length* $\mathrm{wl}(w)$ (with respect to the generating set (a, b)) of a free homotopy class of curves is the total number of letters occurring in the corresponding reduced cyclic word.

Consider a hyperbolic metric (that is, a metric with constant curvature -1) on the pair of pants such that each of the boundary components is a geodesic. It is well known (see, for instance, [Buser 92, Theorem 3.1.7]) that such a hyperbolic metric on the pair of pants is

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determined by the lengths A, B, and C of the three geodesics at each of the boundary components. After A, B, and C have been fixed, every free homotopy class of curves w can be associated with a *geometric length* $G_{A,B,C}(w)$, the length of the unique geodesic in that class.

The goal of this note is to study how the geometric length $G_{A,B,C}(w)$ varies over the population $\mathcal{F}_L$ of all reduced cyclic words w of word length L for each positive integer L. (The set $\mathcal{F}_L$ is endowed with the uniform measure.)

We choose a few representative metrics (mainly, for which the ratio between boundary components is about the same or very different). Using the parameterization described in Section 2, we computed the lengths of a sample of 100 000 words of 100 letters and studied the distribution.

The outcome of our experiments led us to formulate the following conjecture. (See Figure 1).[1]

Conjecture 1.1. *For each hyperbolic metric on the pair of pants with geodesic boundary and such that that length of the boundary components are A, B, and C, there exist constants $\kappa = \kappa(A,B,C)$ and $\sigma = \sigma(A,B,C)$ such that if a reduced cyclic word is chosen at random from among all classes in $\mathcal{F}_L$, then for large L, the distribution of the geometric length approaches the Gaussian distribution with mean $\kappa \cdot L$ and standard deviation $\sigma \cdot L$.*

More precisely, there exist constants $\kappa = \kappa(A,B,C)$ and $\sigma = \sigma(A,B,C)$ such that for every $a < b$, the proportion of words w in $\mathcal{F}_L$ such that

$$\frac{G_{A,B,C}(w) - \kappa L}{\sqrt{L}} \in [a,b]$$

converges, as L goes to infinity, to

$$\frac{1}{\sigma\sqrt{2\pi}} \int_a^b \mathrm{e}^{-x^2/2\sigma^2}\, dx.$$

After the experiment was performed, our student Matt Wrotten found a very elegant proof for the case of closed surfaces that will be part of his doctoral thesis. Danny Calegari pointed out that the ideas used to prove [Calegari 11, Corollary 3.7.5] might be useful in giving a proof of the general case. The referee kindly provided an outline of a proof for the general case in terms of Ruelle operators.

[1]The data on which the histograms are based can be found at http://www.math.sunysb.edu/~moira/HyperbolicLengthData/.

Remark 1.2. The histogram of the sample $(0.1, 1, 10)$ in Figure 1 does not suggest a Gaussian distribution. The peaks in the graph are due to the following: The length of C of one of the boundary components is very large compared to A and B. Therefore, the length of a cyclic reduced word w will be roughly $C/2$ times the number of occurrences of ab, ba, $a\overline{b}$, $\overline{b}a$, $\overline{a}b$, $b\overline{a}$, $\overline{a}\overline{b}$, and $\overline{b}\overline{a}$ in w.

Since it is not computationally feasible to study very long words with the metric $(0.1, 1, 10)$, we study instead the metric $(1, 1, 5)$. When L is small compared with C (14 and 20), we still see peaks in the histogram, approximately at multiples of $C/2 = 2.5$. The peaks start to disappear when L is 50 and do not appear at all when L is 100 (see Figure 2). This suggests that the peak will gradually disappear as the word length goes to infinity.

On the other hand, the difference between the histograms of the metrics $(1, 10, 0.1)$ and $(0.1, 1, 10)$ comes from the fact that the choice of basis for the fundamental group of the pair of pants, although natural, is not symmetric, in the sense that we arbitrarily chose two of the three boundary components.

This type of experimental study was initiated in [Chas 10], where the author computed the distribution of the self-intersection number of free homotopy classes, sampling by word length. It was proved in [Chas and Lalley 12] that this distribution, properly normalized, approaches a Gaussian distribution as the word length goes to infinity.

One can also think of our experiments as a certain product of random matrices.

During the past decades, there have been many results about the limiting behavior of products of random matrices. The first examination of this type of question was [Bellman 54]. In [Furstenberg 63] and [Furstenberg and Kesten 60], the authors stated a central limit theorem under certain hypotheses. Recently, [Pollicott and Sharp 10] studied central limit theorems and their generalizations for matrix groups acting cocompactly or convex cocompactly on the hyperbolic plane. It is likely that their results about infinite products can be adapted to prove our conjecture, which concerns finite products.

2. A PARAMETERIZATION OF ALL HYPERBOLIC METRICS ON THE PAIR OF PANTS WITH GEODESIC BOUNDARY

Let a and b be generators as above of the fundamental group G of a pair of pants that has been endowed with a hyperbolic metric. We can regard G as a subgroup of

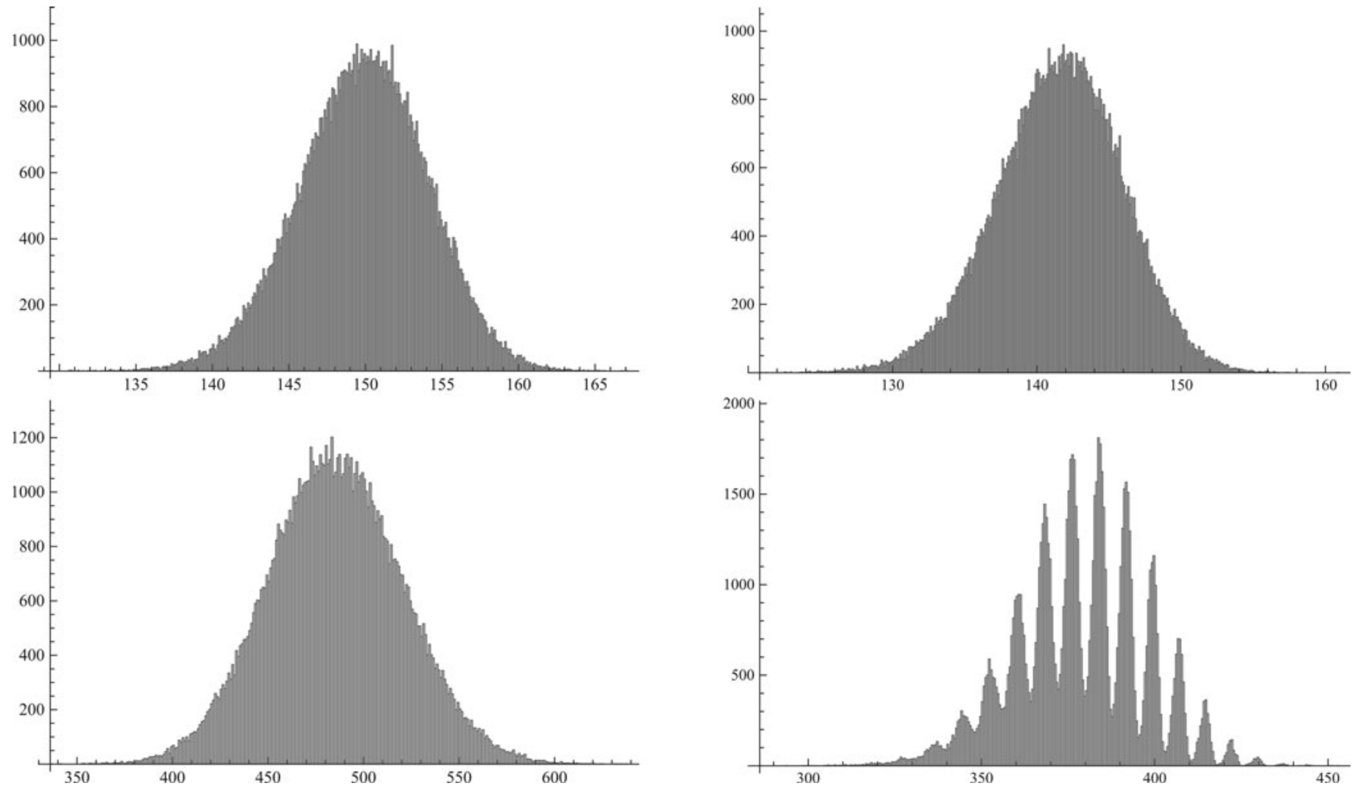

FIGURE 1. Histograms of the geometric length of a sample of 100 000 words of word length 100. The parameters are (A, B, C); $(1, 1, 1)$ top left, $(0.1, 1, 1)$ top right, $(1, 10, 0.1)$ bottom right; $(0.1, 1, 10)$ bottom left.

$\mathrm{PSL}(2, \mathbb{R})$, acting on the upper half-plane. There is then a unique correspondence between closed geodesics on the pair of pants and nontrivial elements of G, where the length l of a closed geodesic is related to the absolute value of the trace of the corresponding element $g \in G$, by

$$\cosh\left(\frac{l}{2}\right) = \frac{|\operatorname{tr}(g)|}{2}.$$

The group G is defined by the hyperbolic metric only up to conjugation in $\mathrm{PSL}(2, \mathbb{R})$. We normalize G so that first, the line orthogonal to the axes of a and b is the imaginary axis, where 0 is closer to the axis of a than to the axis of b. If necessary, we replace a and/or b by its inverse, so that $a(0) > 0$ and $b(0) < 0$. We then normalize further so that a can be realized as reflection in the imaginary axis followed by reflection in the hyperbolic line whose endpoints are at α and 1, $\alpha < 1$. With this normalization, b can be realized as reflection in the hyperbolic line whose endpoints are at β and γ, $\beta < \gamma$, followed by reflection in the imaginary axis.

Since the product $a \cdot b$ is necessarily hyperbolic, we note that we have the inequalities

$$0 < \alpha < 1 < \beta < \gamma. \tag{2–1}$$

Conversely, if α, β, and γ satisfy the above inequality, and G is the group generated by a and b, where a is defined as reflection in the imaginary axis followed by reflection in the line with endpoints α and 1, and b is defined as reflection in the line whose endpoints are at β and γ, followed by reflection in the imaginary axis, then G is necessarily a Fuchsian group representing a pair of pants, and a and b represent simple boundary geodesics. Easy computations show that we now have

$$a = \frac{1}{1-\alpha}\begin{pmatrix} 1+\alpha & 2\alpha \\ 2 & 1+\alpha \end{pmatrix}$$

and

$$b = \frac{1}{\gamma-\beta}\begin{pmatrix} \beta+\gamma & -2\beta\gamma \\ -2 & \beta+\gamma \end{pmatrix}.$$

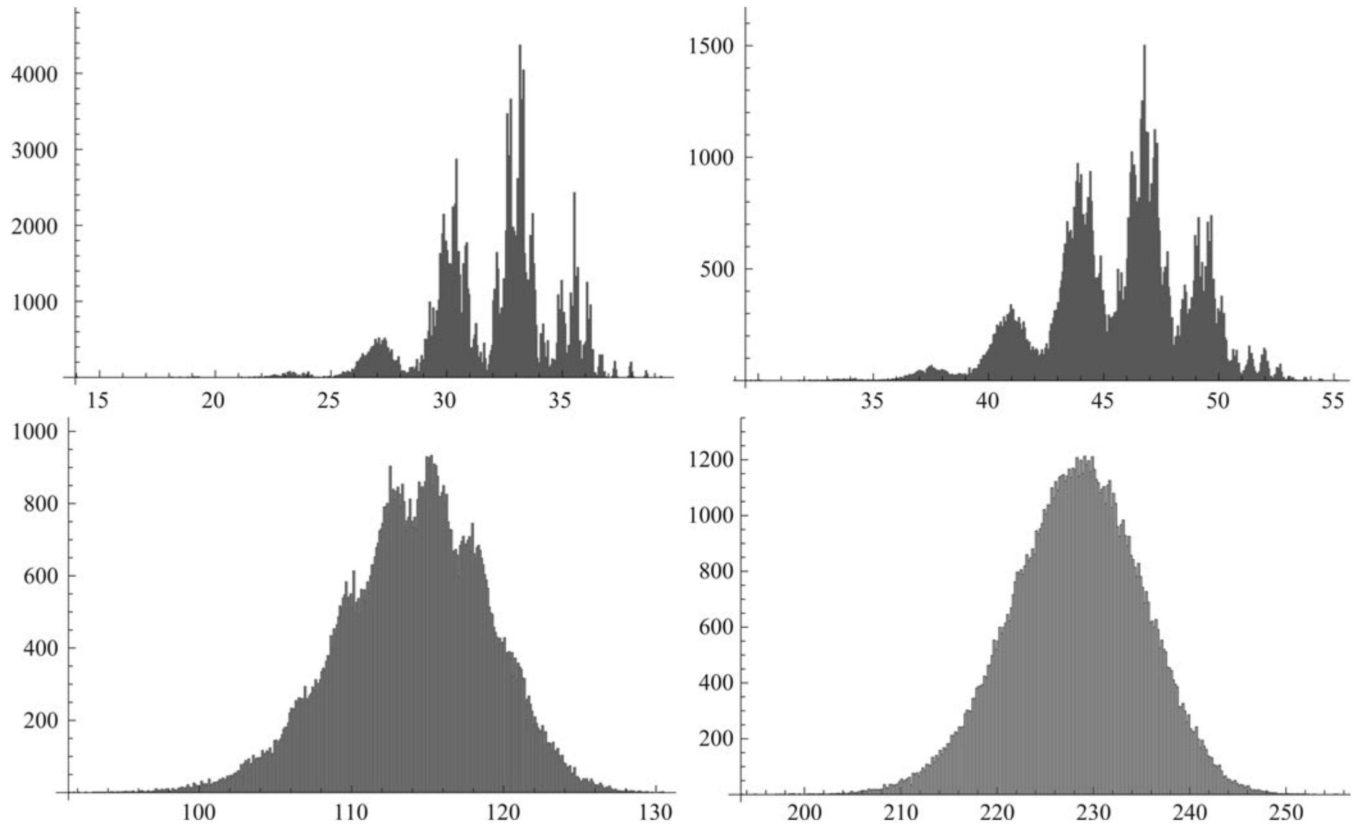

FIGURE 2. Top left: Histogram of all words of word length 14, with metric $(1, 1, 5)$. Top right, bottom left, and bottom right respectively, are histograms of the geometric length of a sample of 100 000 words with parameters $(1, 1, 5)$ and word lengths 20, 50, and 100 respectively.

As above, let A, B, and C denote the lengths of the geodesics corresponding to a, b, and $c = ab$; these are the lengths of the three boundary geodesics on the pair of pants.

Set

$$x = \cosh\frac{A}{2} = \frac{|\mathrm{tr(a)}|}{2} = \frac{1+\alpha}{1-\alpha},$$

$$y = \cosh\frac{B}{2} = \frac{|\mathrm{tr(b)}|}{2} = \frac{\gamma+\beta}{\gamma-\beta},$$

$$z = \cosh\frac{C}{2} = \frac{\mathrm{tr(a \cdot b)}}{2} = -\frac{(1+\alpha)(\beta+\gamma) - 2(\alpha+\beta\gamma)}{(1-\alpha)(\gamma-\beta)}.$$

Using inequality (2–1), the above can be uniquely solved for α, β, and γ. We obtain

$$\alpha = \frac{x-1}{x+1},$$

$$\beta = \frac{(xy+z) + \sqrt{x^2+y^2+z^2+2xyz-1}}{(x+1)(y+1)},$$

$$\gamma = \frac{(xy+z) + \sqrt{x^2+y^2+z^2+2xyz-1}}{(x+1)(y-1)}.$$

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