MAT 364 Extra homework 2

Consider a set I. For each $\mu \in I$, let A_μ be a subset of a given set S. We call I an *indexing set* and the collection of subsets of S indexed by the elements of I is called the *indexed family* of subsets of S. We denote this indexed family by $\{A_\alpha\}_{\mu \in I}$.

If $\{A_\alpha\}_{\alpha \in I}$ is an indexed family, the union $\cup_{\mu \in I} A_\alpha$ is the set of all elements $x \in S$ such that for at least one index, ν, $x \in A_\nu$. Similarly, the intersection $\cap_{\alpha \in I} A_\alpha$ is the set of all elements $x \in S$ such that , $x \in A_\nu$ for all $\nu \in I$.

Consider a set S, three subsets of S, A, B and C and an indexed family $\cup_{\alpha \in I} A_\alpha$ of subsets of S . Prove the following

(1) $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$.

(2) $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$.

(3) $A \subset B$ if and only if $B = A \cup B$.

(4) $A \subset B$ if and only if $A = A \cap B$.

(5) $A \subset S \setminus B$ if and only if $B \subset S \setminus A$.

(6) If $A \subset B \subset C$ then $C \setminus (B \setminus A) = A \cap (C \setminus B)$.

(7) $\cap_{\alpha \in I}(A_\alpha \cap D) = (\cup_{\alpha \in I} A_\alpha) \cap D$.

(8) $S \setminus (\cap_{\alpha \in I} A_\alpha) = \cup_{\alpha \in I}(S \setminus A_\alpha)$.

(9) $S \setminus (\cup \alpha \in I A_\alpha) = \cap_{\alpha \in I}(S \setminus A_\alpha)$.

(10) Let f be a function from a set D to a set E. Let $G \subset D$ and $H \subset G$.

   (a) Prove that $G \subset f^{-1}(f(G))$.
   (b) Prove that $f(G \cup H) = f(G) \cup f(H)$.
   (c) Prove that $f(G \cap H) \subset f(G) \cap f(H)$ and find an example where $f(G \cap H) \neq f(G) \cup f(H)$
   (d) Prove that $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$
   (e) Prove that $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$

Topology review problems

(i) Can a set not have limit points? (prove or disprove)

(ii) Can an infinite set not have limit points?(prove or disprove)

(iii) Prove or disprove: If U is an open set, then $u = Int(Cl(U))$.

(iv) Consider two topological spaces X and Y. Show that the sequence $\{(x_n, y_n\}_{n \in Z_{\geq 0}}$ of elements of $X \times Y$ converges to $(x, y) \in X \times Y$ if and only if $\{(x_n\}_{n \in Z_{\geq 0}}$ converges to x in X and $\{(y_n\}_{n \in Z_{\geq 0}}$ converges to y in Y.

(v) Consider a metric space X and a subset A of X.

   (a) An *isolated point of* A is a point $a \in A$ which has a neighborhood that does not contains other points of A other than a. Prove that every point of A is either a limit point or an isolated point.
   (b) Prove that a sequence in X converges to at most one point.