MAT 364 Extra homework 2

Consider a set $I$. For each $\mu \in I$, let $A_\mu$ be a subset of a given set $S$. We call $I$ an indexing set and the collection of subsets of $S$ indexed by the elements of $I$ is called the indexed family of subsets of $S$. We denote this indexed family by $\{A_\mu\}_{\mu \in I}$.

If $\{A_\mu\}_{\alpha \in I}$ is an indexed family, the union $\cup_{\mu \in I} A_\mu$ is the set of all elements $x \in S$ such that for at least one index, $\nu, x \in A_\nu$. Similarly, the intersection $\cap_{\alpha \in I} A_\alpha$ is the set of all elements $x \in S$ such that , $x \in A_\nu$ for all $\nu \in I$.

Consider a set $S$, three subsets of $S$, $A$, $B$ and $C$ and an indexed family $\cup_{\alpha \in I} A_\alpha$ of subsets of $S$. Prove the following

1. $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$.
2. $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$.
3. $A \subseteq B$ if and only if $B = A \cup B$.
4. $A \subseteq B$ if and only if $A = A \cap B$.
5. $A \subseteq S \setminus B$ if and only if $B \subseteq S \setminus A$.
6. If $A \subseteq B \subseteq C$ then $C \setminus (B \setminus A) = A \cap (C \setminus B)$.
7. $\cap_{\alpha \in I} (A_\alpha \cap D) = (\cup_{\alpha \in I} A_\alpha) \cap D$.
8. $S \setminus (\cap_{\alpha \in I} A_\alpha) = \cup_{\alpha \in I} (S \setminus A_\alpha)$.
9. $S \setminus (\cup_{\alpha \in I} A_\alpha) = \cap_{\alpha \in I} (S \setminus A_\alpha)$.

10. Let $f$ be a function from a set $D$ to a set $E$. Let $G \subseteq D$ and $H \subseteq G$.

   a. Prove that $G \subseteq f^{-1}(f(G))$.
   b. Prove that $f(G \cup H) = f(G) \cup f(H)$.
   c. Prove that $f(G \cap H) \subseteq f(G) \cap f(H)$ and find an example where $f(G \cap H) \neq f(G) \cap f(H)$
   d. Prove that $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$
   e. Prove that $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$

Topology review problems

(i) Can a set not have limit points? (prove or disprove)

(ii) Can an infinite set not have limit points? (prove or disprove)

(iii) Prove or disprove: If $U$ is an open set, then $u = \text{Int}(\text{Cl}(U))$.

(iv) Consider two topological spaces $X$ and $Y$. Show that the sequence $\{(x_n, y_n)_{n \in \mathbb{Z}_{\geq 0}}\}$ of elements of $X \times Y$ converges to $(x, y) \in X \times Y$ if and only if $\{x_n\}_{n \in \mathbb{Z}_{\geq 0}}$ converges to $x$ in $X$ and $\{y_n\}_{n \in \mathbb{Z}_{\geq 0}}$ converges to $y$ in $Y$.

(v) Consider a metric space $X$ and a subset $A$ of $X$.

   a. An isolated point of $A$ is a point $a \in A$ which has a neighborhood that does not contains other points of $A$ other than $a$. Prove that every point of $A$ is either a limit point or an isolated point.
   b. Prove that a sequence in $X$ converges to at most one point.