A relation $R$ on a set $X$ is a subset of ordered pairs of elements of $X$, that is $R \subseteq X \times X$.

A relation $R$ is reflexive if $(x, x)$ belongs to $R$ for all $x$ in $X$.

An example of a relation that is not reflexive is the following: $X$ is a set of people, $R_1$ is the set of pairs $(x, y)$ of people in $X$ such that $x$ is a brother or sister of $y$.

A relation $R$ is symmetric if and only if the following holds: If a pair $(x, y)$ belongs to $R$ then the pair $(y, x)$ belongs to $R$.

A relation $R$ is transitive if and only if the following holds: If the pairs $(x, y)$ and $(y, z)$ belong to $R$ then the pair $(x, z)$ belongs to $R$.

An example of a relation that is reflexive, symmetric and transitive is the following: $X = \mathbb{Z}$, $(x, y) \in R_2$ if $x$ and $y$ have the same parity.

Recall that a relation $R$ is called an equivalence relation if and only if it is symmetric, reflexive and transitive.

If $R$ is an equivalence relation on a set $A$, and $a$ is an element of $A$, the equivalence class of $a$ is the set $\{b \in A \mid (a, b) \in R\}$

A partition of a set $A$ is a disjoint collection of nonempty subsets of $A$ whose union is the whole $A$. In other words, it is a list of subsets $X_\alpha$ of $X$, for $\alpha$ is some set $L$ of labels so that for every $x$ in $X$ there is one and only one label $\alpha$ so that $x$ belongs to $X_\alpha$.

Consequence: There is a function $p$ from $X$ to $L$, namely, $p(x) = \alpha$, where $x \in X_\alpha$.

Workhouse philosophy: In Mathematics, wide use is made of the idea to try to take interesting properties or structures on $X$ and induce them on $L$ using $p$.

A function $f$ from a set $X$ to a set $Y$, is a correspondence that assigns to each element $x$ of $X$, a unique element of $f(x)$ of $Y$.

We can think of a function from a space $X$ to itself, as a relation $R$ which satisfies that for all $x$ in $X$, there is one and only one $y \in X$ such that $(x, y) \in R$.

(1) Give an example of a relation on $X$ that is reflexive and an example of a relation that it is not reflexive, in each of the following cases:

(a) $X$ is a set of people.

(b) $X$ is $\mathbb{R}$, the set of real numbers.

(2) Give an example of a relation on $X$ that is not symmetric and an example of a relation that it is symmetric, in each of the following cases:

(a) $X$ is a set of people.

(b) $X$ is $\mathbb{R}$, the set of real numbers.
(3) Give an example of a relation that is transitive and an example of a relation that it is not transitive.

(4) Give an example of a relation that is symmetric and transitive, but not reflexive.

(5) Give an example of a relation that is reflexive but not transitive.

(6) Give three examples of equivalence relations.

(7) Show that an equivalence relation determines a partition.

(8) Show that a partition determines an equivalence relation.

(9) Show the partitions determined by the equivalence relations you found in problem 6 above.

(10) Define two points on $\mathbb{R}^2$ to be equivalent if they have the same $x$ coordinate. Is it an equivalence relation? If the answer is yes, find the partition it determines.

(11) Consider the partition of $\mathbb{R}^2$ defined by all lines parallel to the line $y = x$. Describe (in terms of the coordinates of the points) the equivalence relation this partition determines.

(12) Define two points $(p, q)$ and $(r, s)$ of the plane to be equivalence if $p - q^3 = r - s^2$. Check that this is an equivalence relation and describe the equivalence classes.

(13) Show that if a function is a symmetric relation then it must be bijective.

(14) Show that if a function $F$ is a symmetric and transitive relation then it must be an involution. (An involution is a function that satisfies $F \circ F = \text{Identity}$.)

(15) Show that if a function is reflexive then it must be the identity mapping.

(16) A function between two finite sets can be described by the enumerating the set of pairs, or by a diagram. Set $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$. In the following cases, if possible, give examples (in both forms, the diagram and enumeration). If it is not possible explain why.

(a) An injective function from $A$ to $B$.

(b) An injective function from $B$ to $A$.

(c) An injective function from $A$ to $A$.

(d) A surjective function from $B$ to $A$.

(17) Consider a surjective function $f$ from $A$ to $B$. Define an equivalence relation $R$ on $A$ by setting $(a, a') \in R$ if and only if $f(a) = f(a')$. Show that $R$ is an equivalence relation, and that there is a bijective correspondence between $B$ and the set $A*$ of equivalence classes of $R$. 
