MAT 360 - Geometric Structures
Practice Final

May 15th, 2007

Here is a practice final with solutions. We will work on this problems in class. All the problem of the previous exam and practice exam, as well as most homework problem are good practice.

Show all work to get full credit; a correct answer with incorrect or no justification **will not get credit**.

Cross the work you do not want to be graded.

Each problem will worth between 12 adn 26 points.

(1) Consider the following axiom system: **Undefined terms**: point, line, incidence. **Axioms**

- (i) Given two distinct points, there is a unique line incident with both.
- (ii) There exists at least one line.
- (iii) There are exactly three points in every line.
- (iv) Not all the points are on the same line.
- (v) There exactly one line through two distinct points.
- (vi) There is at least one point on any two distinct lines.

Now you are asked to prove some results in the given order. You can use any of the results stated before the one you are proving (for instance, you can use (a) in the proof of (b), even if you did not prove (a)).

- (a) Prove that two distinct lines have exactly one point in common.
- (b) Prove that there exist exactly seven points
- (c) Prove that there exist exactly seven lines.
- (d) Find a model
- (e) Study which parallel properties hold in this axiom system.
- (f) Is this axiom system consistent?
- (g) Is this axiom system categorical?
- (h) Prove that one of the axioms is independent of the others. (You can choose the independent axiom)

(2) For the given interpretation of the undefined terms decide whether each of the proposed axioms hold. Indicate also which parallel property holds.

(a) Let c be a circle in the Euclidean plane. **Interpretation**: Points are points in c. Lines are arcs of c containing the endpoints. For points A, B and C, $A * B * C$ means A, B and C are in the same arc and that is in an arc B with endpoints A and C **Incidence Axioms, Betweeness Axioms B1, B2 and B3**

(b) Let H be the half plane, $\{(x, y) \in R^2 : y > 0\}$. **Interpretation**: Points are points in H. Lines are circles perpendicular to the line $\{(x, y) \in R^2 : y = 0\}$ or lines of the form $\{(x, y) \in R^2 : y > 0, x = a\}$, where a is a number (vertical "half lines"). **Incidence Axioms, Betweeness Axioms B1, B2 and B3. Extra credit: all Hilbert Axioms.**

(c) Let l be a line in the Euclidean plane. Points are points in l, lines are subsets of three points in l. Incidence is set membership.

(3) Denote by H Hilbert's parallel axiom : For every line l and every point P not in l there is at most one line m through P parallel to l. Denote by O the statement: If a line intersect one of two parallel lines, then intersect the other.

(a) In neutral geometry, H is equivalent to O. In symbols $H \equiv O$.

(b) Is the statement $H \equiv O$ valid in hyperbolic geometry? If so, combine a. with the negation of Hilbert parallel postulate to obtain a new statement. (You are only allowed to use a., the negation of Hilbert parallel postulate and logic.)

(c) Prove that if Hilbert parallel postulate holds then the relation "is parallel to " is transitive.

(4) (a) In Euclidean geometry, prove the following statement: if a pair of triangles have two corresponding angles which are congruent, then the two triangles are similar.

(b) Does the statement in (a) hold in hyperbolic geometry?

(c) Does the statement in (a). hold in neutral geometry?

(5) In neutral geometry,

(a) Prove that every segment has at most one midpoint.

(b) Prove that every segment has a unique perpendicular bisector.

(6) (Extra credit) Prove that every segment has at least one midpoint.

Problem 1.a We argue by RAA. If (a) is not valid, then there exists two distinct lines, l and m such that either l and m do not have any point in common or they have more than one point in common. By Axiom (vi), l and

m have at least one point in common. So l and m have two or more points in common. Let P and Q be two distinct points in l and m. By Axiom (i), there is a unique line n through P and Q. Since m and l are lines through P and Q, then $l = n$ and $m = n$. Then $m = l$, which contradicts the fact that m and l are distinct. So (a) is proved

Problem 1.b

(1) By axiom (ii) there is at least one line. Denote this line by l.

(2) By Axiom (iii), l has exactly three distinct points P, Q and R.

(3) By Axiom (iv), there is a point S not in l.

(4) By Axiom (v) there are three distinct lines:

   (a) l_P through S and P
   (b) l_Q through S and Q
   (c) l_R through S and R

   (Proof that these lines are different to each other: Assume that $l_P = l_Q$ then l_P is a line through P and Q. By axiom v, $l_P = l$. Then S is in l, contradiction. In the same way, we can prove that these three lines are different from l.)

(5) By Axiom (iii) there are exist three points P_1, Q_1 and R_1.

   (a) $P_1 \neq P$ in l_P
   (b) $Q_1 \neq Q$ in l_Q
   (c) $R_1 \neq R$ in l_R

   (These three points P_1, Q_1 and R_1 are different form P,Q, R and S Proof: If for instance, $P_1 = Q$ then l_P and l_Q are lines through Q and S, then by Axiom v, $l_P = l_Q$ contradiction. The other cases are proved in the same way.)

(6) Now we have at least seven points. Suppose there is one point T different from P, Q, R, S, P_1, Q_1 and R_1.

(7) By Axiom v, there is one line l_T through S and T.

(8) By Axiom (vi), l_T intersects l. Suppose the intersection point is P. Then P and S are points in l_T and l_P. By axiom (v) $l_T = l_P$ and so T is equal to P, P_1 or to S, contraction. Analgoulsy, assuming that if the intersection point is Q or R we arrive to a contradiction. So there is exactly seven points.

Problem 1.c

(1) Note that we have four different lines l, l_P, l_Q, l_R,

(2) By axiom v, there is a line l_{P,R_1} through P and R_1,

(3) This line l_{P,R_1} is different from the four lines we already have (Proof: If, for instance, $l_{P,R_1} = l_P$ then $R_1 \in \{P, S, P_1\}$ contradiction) .

(4) By axiom iii l_{P,R_1} contains one more point. By (c), this point has to be in the set $\{Q, R, P_1, Q_1, S\}$.

(5) By (3) and axiom v above, the third point in l_{P,R_1} is not in $\{Q, R, S, P_1\}$. Then this point is Q_1.

(6) Analogously,by axiom v, there is a line l_{Q,R_1} through Q and R_1. By a similar argument as before, we can prove that the third point of this line is P_1

(7) Analogously,by axiom v, there is a line l_{R,Q_1} through Q and R_1. By a similar argument as before, we can prove that the third point of this line is P_1

(8) We have at least seven lines. Each is formed by one of the following triple of points $\{P, Q, R\}$, $\{P, S, P_1\}$, $\{Q, S, Q_1\}$,$\{R, S, R_1\}$, $\{R, P_1, Q_1\}$,$\{Q, P_1, R_1\}$, or $\{P, R_1, Q_1\}$,

(9) If there exist an eighth line, then this line passes through of our seven points. By axiom v, it is equal to one of the seven lines above.

1.d The model consist in: Points $\{P, Q, R, S, P_1, Q_1, R_1\}$. Lines: $\{P, Q, R\}$, $\{P, S, P_1\}$, $\{Q, S, Q_1\}$,$\{R, S, R_1\}$, $\{R, P_1, Q_1\}$,$\{Q, P_1, R_1\}$, and $\{P, R_1, Q_1\}$,. Incindence: set membership

1.e By Axiom (vi) every pair of lines intersect. Therefore the are no pair of lines which are parallels. Hence the elliptic parallel property holds.

1.f This axiom system is consistent (that is , one can not derive a contradiction from this set of axioms) because it has a model.

1.g This axiom system is categorical (that is, all the models are isomorphic): In a, b and c, we showed that there are exactly seven points and seven lines, intersected in a determined way. So every model has to have this "shape"

1.h To show that one axiom is independent of the others, it is enough to exhibit an interpretation in which all the axioms but the "independent one' holds. The independent axiom we choose is Axiom iii (all the axioms are independent, I think) . Consider the following interpretation: Points A,B,C, Lines: A,B,B,C, A,C. From direct observation we see that all the axioms with the exception of Axiom iii are valid statements int his model.

2.a Axiom B1. holds (follows directly from the interpretation). Axiom B2. Does not hold. Here is a counterexample. Let a be an arc of c with endpoints A and B. Let C be a point such that $A * C * B$. There is no point D in a such that $C * B * D$ because B is the "last" point of a. Axiom B3: Holds because it holds in the Euclidean plane. Axiom IA1 does not hold becasue there are many different arcs c-ontaining a pair of points. Axiom IA2 holds because any arc contains infinitely many points. Axiom IA3 does not holds because any three points are contained an arc.

3.a Assume H. Let us prove O. Let l and m be two parallel lines. Let t be a line that intersects l. Denote the intersection point by Q. Since l is a line parallel to m through P, by H there is no other line parallel to m through P. In particular, since t passes through P, t is not parallel to m. Then by definition of parallel, t intersects m.

Now assume O. Let l be a line and P be a point not in l. If there is no parallel through P to l then H holds. Assume that there is one parallel m to l through P. Let n be a line through P different from m. Since n intersects m (at P), by O, n intersects any line parallel to m. Then n intersects l. IN other words any line through P different from m intersects l. Hence, m is the unique parallel to l through P.

3.b Every statement that can be proven in neutral geometry, holds in hyperbolic geometry.In particular the equivalence proven in 3.a holds in hyperbolic geometry.

By the equivalence $H \equiv O$, since the negation of H, is one of the axioms of hyperbolic geometry, then we have that the negation of O is a theorem of hyperbolic geometry. Hence, a "new" statement is: There exists three lines l, m and n such that l and m are parallel, n intersects l but does not intersect m.

3.c We need to show that if H holds and l, m and n are three lines, such that l is parallel to m, and m is parallel to n then l is parallel to n.

So, assume that l is parallel to m, and m is parallel to n. We argue by RAA. If l and n are not parallel then there is a point P in the intersection of l and n. Hence there are two distinct lines through P parallel to m. This contradicts H and so the result is proven.

4.a Let $\triangle ABC$ and $\triangle DEF$ be two triangles and assume that $A \cong D$ and $B \cong F$.

By Pr.11, the sum of the interior angle of a triangle is $180\circ$. Then the measure in degrees of C equals equals $180\circ-(\sphericalangle A)\circ-\sphericalangle B)\circ$. Analogously, the measure in degrees of E equals $180\circ-(\sphericalangle D)\circ-\sphericalangle F)\circ$. By Theorem 4.3, if two angles are congruent then they have equal measure. Thus $(\sphericalangle A)\circ = (\sphericalangle D)\circ$ and $(\sphericalangle B)\circ = (\sphericalangle E)\circ$. So, $180 \circ -(\sphericalangle D) - \sphericalangle F) = 180 \circ -(\sphericalangle A) \circ -\sphericalangle B)\circ$. By theorem 4.3, $C \cong F$. Since corresponding pairs of angles are congruent, then the triangles are similar.

4.b The statement in a. does not hold in hyperbolic geometry. Here is a counterexample (there might be an easier one. Let me know if you find it). Consider a triangle $\triangle ABC$. Let D be a point between A and B. By

Axiom C4, there exist a ray $\overrightarrow{DE}$ on the same side of $\overleftrightarrow{AB}$ as C such that $\sphericalangle ABC \cong \sphericalangle ADE$.

We claim now that the line $\overleftrightarrow{DE}$ is parallel to the line $\overleftrightarrow{BC}$. Let F be a point on $\overleftrightarrow{DE}$ such that $E * D * F$ (the existence of such a point in guaranteed by Axiom B2. By P.315, $\sphericalangle FDB \cong \sphericalangle ADE$. By T4.1, the lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$ are parallel.

By P3.9, since $\overrightarrow{DE}$ does not intersect $\overleftrightarrow{BC}$ then $\overrightarrow{DE}$ intersects segment AC. Let P denote the intersection point. The quadrialeral $\square PDBC$ is convex. The sum of the interior angles of $\square PDBC$ is

$$(\sphericalangle B)\circ+(\sphericalangle PDB)\circ+(\sphericalangle C)\circ+(\sphericalangle CPD)\circ = 180\circ+(\sphericalangle C)\circ+(\sphericalangle CPD) < 360\circ$$

(the equality follows from T4.3) Then

$$(\sphericalangle C)\circ+(\sphericalangle CPD) < 180\circ$$

By T.4.3,since

$$(\sphericalangle BPA)\circ+(\sphericalangle CPD) = 180\circ$$

then C cannot be congruent to $(\sphericalangle BPA)$

4.c If the statemtn in a were valid in neutral geometry then it would be valid in hypebolic geometry, contradicting b..

5.a Let AB be a segment. We argue by RAA. Suppose there are two midpoints of AB. Denote them by C and D. By definition of midpoint, $AC \cong CB$ and $AD \cong DB$.

By B1, either $A * C * D$ or $A * D * C$. We assume $A * D * C$ (the other case is similar).

By T4.3, $\overline{AB} = \overline{AC} + \overline{CA} = 2\overline{AC} = 2(\overline{AD} + \overline{DC})$.

Also $\overline{AB} = \overline{AD} + \overline{CA} = 2\overline{AD}$.

Thus, $\overline{DC} = 0$. But this contradicts T4.3.7.

5.b Given segment AB. Let l be a perpendicular bisector. The line l intersects AB in the midpoint M, and by the previous problem we knwo that this midpoint is unique.

By Axiom C4, there is a unique ray MP on a given side of the line $\overleftrightarrow{AB}$ such that the angle $\sphericalangle PMA$ is a right angle. This shows that the line $\overleftrightarrow{MP}$ is the unique bisector of AB.

6 There is a proof in the book (in the problems)