References

1. J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 6th ed., McGraw-Hill, 1996.
2. E. S. Loomis, *The Pythagorean Proposition*, National Council of Teachers of Mathematics, 1968.
3. O. Shisha, On Ptolemy's theorem, *Internat. J. Math. & Math. Sci.*, **14** (1991), 410.

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Parity and Primality of Catalan Numbers

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Catalan numbers, like Fibonacci and Lucas numbers, appear in a variety of situations, including the enumeration of triangulations of convex polygons, well-formed sequences of parentheses, binary trees, and the ballot problem [**1**]–[**5**]. Like the other families, Catalan numbers are a great source of pleasure, and are excellent candidates for exploration, experimentation, and conjecturing.

They are named after the Belgian mathematician Eugene Catalan (1814–1894), who discovered them in his study of well-formed sequences of parentheses. However, Leonhard Euler (1707–1783) had found them fifty years earlier while counting the number of triangulations of convex polygons [**3**]. But the credit for the earliest known discovery goes to the Chinese mathematician Antu Ming (ca. 1692–1763), who was aware of them as early as 1730 [**6**].

In 1759 the German mathematician and physicist Johann Andreas von Segner (1707–1777), a contemporary of Euler, found that the number C_n of triangulations of a convex polygon satisfies the recursive formula

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0, \tag{1}$$

where $C_0 = 1$ [**3, 5**]. The numbers C_n are now called *Catalan numbers*. It follows from (1) that $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, and so on.

Using generating functions and Segner's formula, an explicit formula for C_n can be developed [**5**]:

$$\begin{aligned} C_n &= \frac{(2n)!}{(n+1)!n!} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned} \tag{2}$$

Consequently, C_n can be extracted from Pascal's triangle by dividing the central binomial coefficient $\binom{2n}{n}$ in row $2n$ by $n+1$.

In this note, we identify Catalan numbers that are odd, and those that are prime. In Table 1, which gives the first eighteen Catalan numbers, those that are odd are marked with asterisks and those that are prime with daggers.

Parity of Catalan Numbers. It follows from the table that when $n \le 17$, C_n is odd for $n = 0, 1, 3, 7,$ and 15, all of which are of the form $2^m - 1$. When $m > 0$, such numbers are known as Mersenne numbers.

Theorem. *For $n > 0$, C_n is odd if and only if n is a Mersenne number.*

Table 1. The first 18 Catalan Numbers.

n	0	1	2	3	4	5	6	7	8
C_n	1	1*	$2^{\dagger}$	$5^{*\dagger}$	14	42	132	429*	1430

n	9	10	11	12	13	14	15	16	17
C_n	4862	16796	58786	208012	742900	2674440	9694845*	35357670	129644790

Proof. It follows from the recurrence relation (1) that

$$C_n = \begin{cases} 2(C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{(n/2)-1}C_{n/2}) & \text{if } n \text{ is even} \\ 2(C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{(n-3)/2}C_{(n+1)/2}) + C^2_{(n-1)/2} & \text{otherwise.} \end{cases}$$

Consequently, for $n > 0$, C_n is odd if and only if both n and $C_{(n-1)/2}$ are odd. The same argument implies that C_n is odd if and only if $(n-1)/2$ and $C_{(n-3)/4}$ are both odd or $(n-1)/2 = 0$. Continuing this finite descent, it follows that C_n is odd if and only if $C_{n-(2^m-1)/2^m}$ is odd, where $m \geq 1$. But the least value of k for which C_k is odd is $k = 0$. Thus the sequence of these *if and only if* statements terminates when $n - (2^m - 1)/2^m = 0$; that is, when $n = 2^m - 1$, a Mersenne number. ∎

Primality of Catalan Numbers. Returning to Table 1, we make another observation: Exactly two of these Catalan numbers are prime. The following theorem confirms this.

Theorem. The only prime Catalan numbers are C_2 and C_3.

Proof. It follows from (2) that $(n+2)C_{n+1} = (4n+2)C_n$. Assume that C_n is a prime for some n. It follows from (1) that if $n > 3$, then $(n+2)/C_n < 1$; so $C_n > n+2$. Consequently, $C_n \mid C_{n+1}$, so $C_{n+1} = kC_n$ for some positive integer k. Then $4n+2 = k(n+2)$, whence $1 \leq k \leq 3$ and thus $n \leq 4$. It follows that C_2 and C_3 are the only Catalan numbers that are prime. ∎

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References

1. D.I.A. Cohen, *Basic Techniques of Combinatorial Theory*, Wiley, 1978.
2. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd ed., Wiley, 1968.
3. M. Gardner, Catalan Numbers: An Integer sequence that Materializes in Unexpected Places, *Scientific American*, 234 (1976) 120–125.
4. R. K. Guy, The Second Strong Law of Small Numbers, *Mathematics Magazine*, 63 (1990) 3–20.
5. T. Koshy, *Discrete Mathematics with Applications*, Elsevier Academic Press, 2004.
6. P. J. Larcombe, The 18th Century Chinese Discovery of the Catalan Numbers, *Mathematical Spectrum*, 32 (1999) 5–7.