Parity and Primality of Catalan Numbers
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References

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Catalan numbers, like Fibonacci and Lucas numbers, appear in a variety of situations, including the enumeration of triangulations of convex polygons, well-formed sequences of parentheses, binary trees, and the ballot problem [1]–[5]. Like the other families, Catalan numbers are a great source of pleasure, and are excellent candidates for exploration, experimentation, and conjecturing.

They are named after the Belgian mathematician Eugene Catalan (1814–1894), who discovered them in his study of well-formed sequences of parentheses. However, Leonard Euler (1707–1783) had found them fifty years earlier while counting the number of triangulations of convex polygons [3]. But the credit for the earliest known discovery goes to the Chinese mathematician Antu Ming (ca. 1692–1763), who was aware of them as early as 1730 [6].

In 1759 the German mathematician and physicist Johann Andreas von Segner (1707–1777), a contemporary of Euler, found that the number \( C_n \) of triangulations of a convex polygon satisfies the recursive formula

\[
C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0,
\]

where \( C_0 = 1 \) [3, 5]. The numbers \( C_n \) are now called Catalan numbers. It follows from (1) that \( C_1 = 1, C_2 = 2, C_3 = 5 \), and so on.

Using generating functions and Segner’s formula, an explicit formula for \( C_n \) can be developed [5]:

\[
C_n = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1} \binom{2n}{n}.
\]

Consequently, \( C_n \) can be extracted from Pascal’s triangle by dividing the central binomial coefficient \( \binom{2n}{n} \) in row \( 2n \) by \( n + 1 \).

In this note, we identify Catalan numbers that are odd, and those that are prime. In Table 1, which gives the first eighteen Catalan numbers, those that are odd are marked with asterisks and those that are prime with daggers.

Parity of Catalan Numbers. It follows from the table that when \( n \leq 17 \), \( C_n \) is odd for \( n = 0, 1, 3, 7, \) and \( 15 \), all of which are of the form \( 2^m - 1 \). When \( m > 0 \), such numbers are known as Mersenne numbers.

Theorem. For \( n > 0 \), \( C_n \) is odd if and only if \( n \) is a Mersenne number.
Table 1. The first 18 Catalan Numbers.

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>C_n</td>
<td>1</td>
<td>1*</td>
<td>2†</td>
<td>5†</td>
<td>14</td>
<td>42</td>
<td>132</td>
<td>429*</td>
<td>1430</td>
</tr>
<tr>
<td>n</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>C_n</td>
<td>4862</td>
<td>16796</td>
<td>58786</td>
<td>208012</td>
<td>742900</td>
<td>2674440</td>
<td>9694845*</td>
<td>35357670</td>
<td>129644790</td>
</tr>
</tbody>
</table>

Proof. It follows from the recurrence relation (1) that
\[
C_n = \begin{cases} 
2(C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{(n/2)-1}C_{n/2}) & \text{if } n \text{ is even} \\
2(C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{(n-3)/2}C_{(n+1)/2}) + C_{(n-1)/2}^2 & \text{otherwise.}
\end{cases}
\]

Consequently, for \( n > 0 \), \( C_n \) is odd if and only if both \( n \) and \( C_{(n-1)/2} \) are odd. The same argument implies that \( C_n \) is odd if and only if \( (n - 1)/2 \) and \( C_{(n-3)/4} \) are both odd or \( (n - 1)/2 = 0 \). Continuing this finite descent, it follows that \( C_n \) is odd if and only if \( C_{n-2m} = 2m \) is odd, where \( m \geq 1 \). But the least value of \( k \) for which \( C_k \) is odd is \( k = 0 \). Thus the sequence of these if and only if statements terminates when \( n - 2m = 1 \); that is, when \( n = 2^m - 1 \), a Mersenne number.

Primality of Catalan Numbers. Returning to Table 1, we make another observation: Exactly two of these Catalan numbers are prime. The following theorem confirms this.

Theorem. The only prime Catalan numbers are \( C_2 \) and \( C_3 \).

Proof. It follows from (2) that \( (n + 2)C_{n+1} = (4n + 2)C_n \). Assume that \( C_n \) is a prime for some \( n \). It follows from (1) that if \( n > 3 \), then \( (n + 2)/C_n < 1 \); so \( C_n > n + 2 \). Consequently, \( C_n \mid C_{n+1} \), so \( C_{n+1} = kC_n \) for some positive integer \( k \). Then \( 4n + 2 = k(n + 2) \), whence \( 1 \leq k \leq 3 \) and thus \( n \leq 4 \). It follows that \( C_2 \) and \( C_3 \) are the only Catalan numbers that are prime.

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References