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Source: *The American Mathematical Monthly*, Vol. 104, No. 5 (May, 1997), pp. 445-446
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2974739
Accessed: 24/03/2010 21:25

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and thus equation (2) simplifies to

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \pmod{p}. \tag{3}$$

In the sum on the left we replace k by $p - k \equiv -k \pmod{p}$ and use Fermat's Little Theorem to obtain

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \equiv -2^p \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \equiv -2 \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \pmod{p}.$$

The sum on the right of (3) we rewrite as

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} + \sum_{k=1}^{(p-1)/2} \frac{(-1)^{p-k}}{p-k} \equiv 2 \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} \pmod{p}.$$

This proves (1).

In the literature, congruences of a type similar to (1) can be found; however, in general they are of a much deeper nature. For example, in **[1]** with the help of properties of the Pell sequence $((1+\sqrt{2})^n)_{n \in N}$ it is shown that

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \pmod{p}. \tag{4}$$

It seems unlikely that (4) can be proved with the simple approach we have used here.

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A Geometrical Method for Finding an Explicit Formula for the Greatest Common Divisor

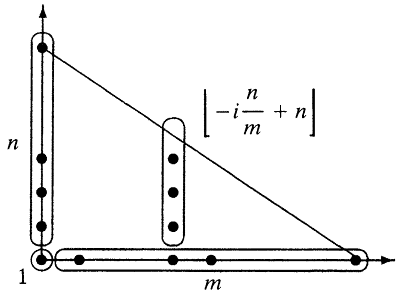
Marcelo Polezzi

This note presents an explicit formula for the greatest common divisor (g.c.d.) of two integers derived using a simple geometrical argument.

In [1], chapter 3, an expression was deduced, from which one can easily obtain a formula for the g.c.d. as a particular case. However, the derivation of that expression is very tiring and lengthy.

Here is the result to be proved:

Theorem. *Let m and n be positive integers. Then g.c.d.$(m, n) = 2\sum_{i=1}^{m-1}\left\lfloor i\frac{n}{m}\right\rfloor + (m + n) - mn$, where $\lfloor x\rfloor$ is the greatest integer less than or equal to x.*



Proof: Consider the triangle $M = \left\{(x, y) \in \mathbb{R}^2 | x \geq 0; y \geq 0; y \leq -\frac{n}{m}x + n\right\}$. We have

$$\#(M \cap \mathbb{Z}^2) = \sum_{i=1}^{m-1}\left\lfloor -i\frac{n}{m} + n\right\rfloor + (m + n + 1)$$
$$= \sum_{i=1}^{m-1}\left\lfloor (m - i)\frac{n}{m}\right\rfloor + (m + n + 1) = \sum_{i=1}^{m-1}\left\lfloor i\frac{n}{m}\right\rfloor + (m + n + 1). \quad (1)$$

On the other hand, by considering the triangle as half of a rectangle, we obtain

$$\#(M \cap \mathbb{Z}^2) = \frac{(m + 1)(n + 1) + (d + 1)}{2} \quad (2)$$

where $d = g.c.d.(m, n)$, since the number of lattice points on the hypotenuse is equal to $(d + 1)$. In fact, let $y_i = -(n/m)x_i + n$. The set of integers x_i between 0 and m such that y_i is an integer is

$$\left\{0, \frac{m}{d}, 2\frac{m}{d}, \ldots, (d - 1)\frac{m}{d}, m\right\}.$$

Hence, equating (1) and (2), we have

$$\sum_{i=1}^{m-1}\left\lfloor i\frac{n}{m}\right\rfloor + (m + n + 1) = \frac{(m + 1)(n + 1) + (d + 1)}{2}.$$

Therefore, $d = 2\sum_{i=1}^{m-1}\left\lfloor i\frac{n}{m}\right\rfloor + (m + n) - mn$.

Remark. The formula clearly holds for $n = 0$.

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