Proof: If neither n nor $n+2$ is of this form then $\phi(n)$ and $\phi(n+2)$ would both be divisible by 4, so that their difference could not be 2.

2. *The case* $\alpha=1$ *leads to the classes of solutions mentioned above.*

Proof: If n is a prime then $\phi(n)=n-1$, while if n is composite then n has a prime factor $\leqq\sqrt{n}$, so that $\phi(n)\leqq n(1-1/\sqrt{n})=n-\sqrt{n}$. Hence if one of n and $n+2$ is prime so is the other. If $n=2p$, then $\phi(n)=p-1=(n-2)/2$, so that $\phi(n)+2=\phi(n+2)$ would imply $\phi(n+2)=(n+2)/2$, in which case $n+2$ is clearly a power of 2. If $n+2=2p'$ then $\phi(n+2)=p'-1=n/2$, so that $\phi(n)=n/2-2$, and n must be of the form $4p$.

3. *We have* $\alpha\neq 2$.

Proof: If $n=p^2$, then $\phi(n)=n-\sqrt{n}$, while if $n+2$ is composite (but clearly not a square), then $\phi(n+2)<(n+2)(1-1/\sqrt{n})^2<\phi(n)+2$. Similarly we can dispose of the cases $n=2p^2$, $n+2=p^2$ and $n+2=2p^2$.

This leaves relatively few numbers $<10^6$ to be examined and these can be tested directly.

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CONGRUENCES FOR SETS OF PRIMES

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1. Introduction. Wilson's function $P_1(n)$ is the function $P_1(n)\equiv(n-1)!+1$. By Wilson's theorem the condition $P_1(n)\equiv 0 \bmod n$ is necessary and sufficient in order that an integer $n>1$ be prime. In this note we find a congruence condition, similar to the above, for twin primality, and we indicate a method which furnishes a condition for sets of prime numbers of any prescribed type.

2. Twin primes. We shall establish the following result:

THEOREM. *A necessary and sufficient condition that two integers, n and $n+2$, $n>1$, both be prime is that*

$$4[(n-1)!+1]+n\equiv 0 \bmod n(n+2). \tag{1}$$

Proof. The sufficiency is obvious as divisions by n and $n+2$ separately reduce either to Wilson's theorem or to a simple modification of it.

The necessity follows as easily, but we wish to indicate how (1) may be obtain directly. Thus, with n and $n+2$ both primes, we have

$$(n-1)!+1\equiv 0 \bmod n, \tag{2}$$

$$(n+1)!+1\equiv 0 \bmod (n+2). \tag{3}$$

Reducing the factorial of (3) mod $(n+2)$ and rewriting as an equation we obtain

$$2[(n-1)!] + 1 = k(n+2), \qquad k \text{ some integer}; \tag{4}$$

then, using (2), we must have

$$2k + 1 \equiv 0 \bmod n. \tag{5}$$

Substitution of (5) in (4) determines the congruence of the theorem.

It may be noted that if 1 is considered the first prime, then the restriction $n>1$ can be deleted from the above theorem.

3. Further congruences. By analogous procedure, now using (1), we find that three positive integers n, $n+2$ and $n+6$, are a prime triple if and only if

$$4320[4(\overline{n-1!} + 1) + n] + 361n(n+2) \equiv 0 \bmod n(n+2)(n+6). \tag{6}$$

As stated, 1 is admitted as the first prime; if desired this may be obviated by requiring $n>1$. A similar congruence may be obtained for the other possible class of prime triples given by integers n, $n+4$, and $n+6$.

We indicate a less laborious method than that of the theorem for obtaining (6). By a modification of Wilson's theorem, $n+6$ is prime if and only if

$$720(n-1)! + 1 \equiv 0 \bmod (n+6). \tag{7}$$

Then using (1) we write

$$A[4(\overline{n-1!} + 1) + n] + Bn(n+2) \equiv 0 \bmod n(n+2)(n+6),$$

and seek integers A and B so that this congruence mod $(n+6)$ reduces to a multiple of (7). This gives (5) immediately, and the process can be applied in this recursive fashion to prime sets of any prescribed type.

4. Prime quadruples. Let $P_2(n)$ be the function on the left of (1), and $P_3(n)$ be the left side of (6). We then have

$$P_2(n) = 4P_1(n) + n,$$

and

$$P_3(n) = 4320P_2(n) + 361n(n+2).$$

The four positive integers n, $n+2$, $n+6$, $n+8$ may each be prime, the set then being a prime quadruple consisting of two sets of twin primes. For the function associated with this set, $P_4(n)$, we find

$$P_4(n) = 224P_3(n) + 111n(n+2)(n+6).$$

The congruence condition

$$P_4(n) \equiv 0 \bmod n(n+2)(n+6)(n+8)$$

is necessary and sufficient for the set to be a prime quadruple. By (1) a like condition is presented by the two congruences

$$P_2(n) \equiv 0 \bmod n(n+2)$$
$$P_2(n+5) \equiv 0 \bmod (n+6)(n+8).$$

As an exercise one might show that these two sets of conditions actually are equivalent.

CLASSROOM NOTES

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LOGARITHMIC INTEGRATION*

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1. Introduction. Logarithmic differentiation is a device by means of which complicated products, quotients, and exponential functions may be differentiated with much less algebraic manipulation than is required by the use of the standard formula. We recall the rule for logarithmic differentiation:

$$\frac{dU(x)}{dx} = U(x)\frac{d}{dx}\ln U(x) \tag{1}$$

Applying this to a numerical example we have:

$$\frac{d}{dx}\frac{\sqrt{x^2+1}}{\sqrt[3]{x^3+1}} = \frac{x-x^2}{\sqrt{x^2+1}\,(x^3+1)^{4/3}}. \tag{2}$$

The integration of the answer, however, cannot be accomplished by standard methods. By integration we mean, as usual, the expression of the integral in a finite number of terms containing only elementary functions. In this note we outline a method for integrating certain expressions of this form, and we call the method "logarithmic integration."

2. Case I. Here we are concerned with a method of integrating certain ex-

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