MAT211: Linear Transformations and isomorphisms

- Linear transformations, image, rank, nullity
- Isomorphism and isomorphic spaces
- Theorem: Coordinate transformations are isomorphisms
- Properties of isomorphisms

Definition
- Consider two linear spaces V and W.
- A function T from V to W is called a linear function if for every pair of elements f and g in V, and every scalar k,
- \( T(f + g) = T(f) + T(g) \)
- \( T(af) = aT(f) \)

EXAMPLE: Find out whether the transformation from \( \mathbb{R}^{2\times2} \) to \( \mathbb{R}^3 \) is linear, where \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)

\[
T(M) = M^2
\]
\[
T(M) = 7M
\]
\[
T(M) = P M P^{-1}
\] where P is

\[
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}
\]

Definitions
- The image of a linear transformation T from V to W, denoted by \( \text{Im } T \), is the subset of W \( \{ T(f) : f \in V \} \).
- The kernel of a linear transformation T from V to W, denoted by \( \ker T \), is the subset of V \( \{ f \in V : T(f) = 0 \} \).
- If the image of a linear transformation T is finite dimensional, then the dimension of \( \text{Im } T \) is called the rank of T.
- If the kernel of a linear transformation T is finite dimensional then the dimension of \( \ker T \) is called nullity of T.

EXAMPLE: Find rank, image, kernel and nullity of the transformation from \( \mathbb{R}^{2\times2} \) to \( \mathbb{R}^3 \), \( T(M) = (d, a, b) \)
where M is

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

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- Find out if the transformations from \( \mathbb{P}_2 \) to \( \mathbb{R} \) defined by \( T(t) = \int_0^1 p(t) dt \) is linear.
- If it is linear, find image, rank, kernel and nullity.
Is the transformation $T(M) = (a, b, c, d)$ from $\mathbb{R}^{2 \times 2}$ to $\mathbb{R}^4$ an isomorphism?  

Note: If there is an isomorphism between $V$ and $W$ then $V$ and $W$ have the same dimension.

Consider the linear transformation $T$ from $P_2$ to $P_1$ given by $T(p(t)) = p'(t)$.

• Is it an isomorphism?
• Find rank, nullity, image and kernel.

**Exercise:** illustrate with examples the above theorem

**Example**

• Find a two different bases of the linear space of $2 \times 2$ matrices and find the coordinate transformation for those bases