A linear (or vector) space \( V \) is a set of elements endowed with two operations
- + addition: for each \( f \) and \( g \) in \( V \), \( f+g \) is an element in \( V \).
- \( \cdot \) multiplication: For each \( f \) in \( V \) and each \( k \) in \( \mathbb{R} \), \( k\cdot f \) is an element in \( V \).

Moreover, these operations satisfy the following properties
- \((f+g)+h = f+(g+h)\)
- \(f+g = g+f\)
- There exists a unique element in \( V \), denoted by \( 0 \) and called the neutral element such that \( f+0 = 0+f = f \)
- For each \( f \) in \( V \) there exists a unique element in \( V \) denoted by \(-f\) such that \( f+(-f)=0 \).
- \( k \cdot (f+g) = k \cdot f + k \cdot g \)
- \( (c+k) \cdot f = c \cdot f + k \cdot f \)
- \( c \cdot (k \cdot f) = (c \cdot k) \cdot f \)
- \( 1 \cdot f = f \)

EXAMPLES of Linear Spaces:
- \( \mathbb{R}^n \).
- The set of all polynomials.
- The set of all polynomials of degree at most two.
- The set of all infinite sequences of real numbers.
- The set of all \( m \times n \) matrices
- The space of all \( 2 \times 2 \) matrices such that \( a+d=0 \)

Questions: In each of the linear spaces
- give examples of + and \( \cdot \).
- What is \( 0 \)?
- If \( v \) is in the linear space, what is \(-v\)?

EXAMPLES
- Is the set of all \( 2 \times 2 \) invertible matrices a subspace of the linear space formed by all \( 2 \times 2 \) matrices?
- Denote by \( P_4 \) the set of all polynomials of degree at most 4. Is \( P_2 \) a subspace of \( P_4 \)?
- Is the subset of all polynomials of degree 2 a subset of \( P_4 \)?
Consider the elements $f_1, f_2, \ldots, f_n$ in a linear space $V$

- $f_1, f_2, \ldots, f_n$ span $V$ if every element in $V$ is a linear combination of the elements $f_1, f_2, \ldots, f_n$.
- $f_i$ is redundant if it is a linear combination of $f_1, f_2, \ldots, f_{i-1}$.
- $f_1, f_2, \ldots, f_n$ are linearly independent if none of them is redundant.
- $f_1, f_2, \ldots, f_n$ form a basis if they are linearly independent and span $V$.

**Example:** Consider the linear space $M$ of all matrices $2 \times 3$.

a. Find elements $f_1, f_2, \ldots, f_n$ in $M$ that span $M$

b. Find a basis of $M$.

c. Can you find a basis of $M$ that does not span?

d. Can you find a subset of $M$ that spans but is not a basis? If so, indicate the redundant vectors.

**Theorem:** If a basis of a vector space has $n$ elements then all basis of a linear space have the $n$ elements.

**Definition:** If a linear space $V$ has a basis with $n$ elements we say that the dimension of $V$ is $n$.

**EXAMPLE:** Find a basis and determine the dimension

- The linear space of all $2 \times 2$ matrices.
- The space of all $2 \times 2$ matrices such that $a+d=0$.
- $P_2$, the space of all polynomials of degree at most 2.
- The space of all polynomials.
- The space of all $2 \times 2$ matrices that commute with
  
  \[
  \begin{bmatrix}
  0 & 1 \\
  1 & 1
  \end{bmatrix}
  \]