MAT 211

- The matrix of a linear transformation
- The B-matrix of a linear transformation
- The columns of the B-matrix of a linear transformation
- Change of basis matrix
- Change of basis in a subspace of $\mathbb{R}^n$
- Change of basis for the matrix of a linear transformation

**Summary of this section:** $A$ and $B$ bases of linear space $V$, $T$ a linear transformation from $V$ to $V$

- Coordinate Transformation from $V$ to $R^n$ $T_A^B = [T]_B^A$
- B-matrix of $T$ is $L_A T L_A^{-1}$
- $B=\{f_1,f_2,...,f_r\}$ then B-matrix of $T$ is $[[T(f_1)]_B, [T(f_2)]_B,...,[T(f_r)]_B]$,
- The matrix $S = S_{B\rightarrow A}$ of the linear transformation $L_A (L_A)^{-1}$ from $R^r$ to $R^r$ is called the **change of basis matrix from $B$ to $A$**.
- The change of basis from $B=\{b_1,b_2,...,b_r\}$ to $A$ is $[[a_1], [a_2],...,[a_r]]_{B\rightarrow A}$.
- If $f$ is in $V$ then $[f]_A = S [f]_B$ where $S$ is the change of basis matrix from $B$ to $A$.
- $A=\{a_1,a_2,...,a_n\}$ and $B=\{b_1,b_2,...,b_n\}$, $S$ the change of basis matrix from $B$ to $A$. Then $[b_1,b_2,...,b_n] = [a_1,a_2,...,a_n]S$.

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**Example**

Consider the space $U$ of upper triangular $2 \times 2$ matrices and the linear transformation $T$ from $U$ to $U$ defined by $T(M) = AM$ where $A$ is

$$
\begin{bmatrix}
1 & -2 \\
0 & 3
\end{bmatrix}
$$

Find a basis $B$ of $U$ and for each element $z$ of $U$, find $[T(z)]_B$.

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Checklist 4.1 and 4.2

- **4.1 Linear spaces.**
  - Definition (when a set is a linear space)
  - Main examples
    - $\mathbb{R}^n$
    - Matrices
    - Polynomials
  - Vectors $f_1,f_2,...,f_r$ in a linear space $V$ have:
    - span
    - linear independent,
  - form a basis.
  - Coordinate transformation
  - Dimension of a linear space

- **4.2 Linear transformations and isomorphisms**
  - Definition (when a function is a linear transformation.)
  - Kernel, image, rank, nullity of a linear transformation (what are they and how to compute them)
  - Rank-nullity theorem
  - Isomorphism
    - Definition
    - Properties (what is the kernel, image, dimension of domain and target space)

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**The B-matrix of a linear transformation**

- Consider a linear transformation $T$ from $V$ to $V$ where $V$ is an $n$-dimensional linear space. Let $B$ denote a basis of $V$.
- The matrix $A$ of the transformation from $R^n$ to $R^n$ defined by $L_A \circ T \circ L_A^{-1}$ is called the **B-matrix of $T$**.
- If $A$ is the B-matrix of $T$ then $A [v]_B = [T(v)]_B$.
- The columns of $B$ are the B-coordinate vectors $[T(b_1)]_B, [T(b_2)]_B,...,[T(b_n)]_B$.

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**The B-matrix of a linear transformation**

- Consider a linear transformation $T$ from $V$ to $V$. Let $B$ be matrix of $T$ with respect to a basis $B=\{b_1,b_2,...,b_n\}$.
- The columns of $B$ are the B-coordinate vectors $[T(b_1)]_B, [T(b_2)]_B,...,[T(b_n)]_B$.
  - Give the matrix of the linear transformation $T(f)=f''+2f'$ from $P_2$ to $P_1$ with respect to the basis $(1,t,t^2)$.
  - Find basis of the kernel and the image and compute rank and nullity of $T$. 

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**Summary of this section:** $A$ and $B$ bases of linear space $V$, $T$ a linear transformation from $V$ to $V$
3. Consider the basis \( A \) and find the kernel, image, rank and nullity.

2. Consider two basis \( A \) and \( B \) of an \( n \)-dimensional vector space \( V \).

1. The matrix \( S = S_{B \to A} \) of the linear transformation \( L \) o (LA)\(^{-1} \) from \( R^n \) to \( R^n \) is called the \textit{change of basis matrix from} \( B \) to \( A \).

- \( S_{B \to A} = \) an invertible matrix and \( S_{B \to A}^{-1} = S_{A \to B} \)
- \[ S_{B \to A} = (S_{B \to A})^{-1} \]

Recall: The coordinate Transformation from \( V \) to \( R^n \) is \( L(f) = [f]_B \).

Recall: The \textit{change of basis matrix from} \( B \) to \( A \) is

\[ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\textit{Example:} In the plane \( V \) defined by the equation \( 2x+y-2z=0 \) consider the basis

1. Find the change of basis matrix from \( B \) to \( A \)
2. Find the change of basis matrix from \( A \) to \( B \)
3. Write an equation relating the matrices \([a_1,a_2,\ldots,n] \) and \([b_1,b_2,\ldots,n] \), where \( A = (a_1,a_2) \) and \( B = (b_1,b_2) \)