MAT211 Lecture 14

- Orthogonal projections and orthogonal basis
- Orthogonality, length, unit vectors
- Orthonormal vectors: definition and properties
- Orthogonal projections: definition, formula and properties.
- Orthogonal complements
- Pythagorean theorem, Cauchy inequality, angle between two vectors

- A vector \( \mathbf{v} \) in \( \mathbb{R}^n \) is orthogonal to a subspace \( V \) of \( \mathbb{R}^n \) if it is orthogonal to all vectors in \( V \).
- If \( \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\} \) is a basis of \( V \), then \( \mathbf{v} \) is orthogonal to \( V \) if (and only if) \( \mathbf{v} \) is orthogonal to \( \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m \).

Example
- Consider the subspace \( V \) of \( \mathbb{R}^3 \) span by \( (1,1,1) \) and \( (1,0,1) \).
- Find all the vectors orthogonal to \( V \).

- Two vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^n \) are perpendicular or orthogonal if \( \mathbf{u} \cdot \mathbf{v} = 0 \).
- The length of a vector \( \mathbf{v} \) in \( \mathbb{R}^n \) is \( ||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \).
- A vector \( \mathbf{v} \) in \( \mathbb{R}^n \) is called a unit vector if \( ||\mathbf{v}|| = 1 \).

Example
- Find a unit vector in the line spanned by \( (1,1,3) \).
- Find a vector of length 2 orthogonal to \( (1,1,3) \).

The vectors \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \) of \( \mathbb{R}^n \) are called orthonormal if they are all unit vectors and are orthogonal to one another. In symbols

\[
(\mathbf{u}_i, \mathbf{u}_j) = 0 \quad \text{if} \quad i \neq j,
\]

and

\[
(\mathbf{u}_i, \mathbf{u}_i) = 1
\]

Example: Find an orthonormal basis of the subspace of \( \mathbb{R}^3 \) of equation \( x+y+z=0 \).
• Orthonormal vectors are linearly independent.
• A set of n orthonormal vectors in \( \mathbb{R}^n \) form a basis.

Consider the vectors \( v_1 = (1/\sqrt{2})(1,0,1), \)
\( v_2 = (0,1,0), \) and \( v_3 = (1/\sqrt{2})(1,0,-1). \)
• Check that \( v_1, v_2 \) and \( v_3 \) are orthonormal.
• Are they linearly independent?

Let \( V \) be a subspace of \( \mathbb{R}^n \) and let \( x \) be a vector \( \mathbb{R}^n. \) Then there exists
unique vectors \( x^\perp \) and \( x^\parallel \) such that
\[
x = x^\parallel + x^\perp
\]
• \( x^\parallel \) in \( V \)
• \( x^\perp \) is orthogonal to \( V \).

If \( V \) is a subspace of \( \mathbb{R}^n \) with orthonormal basis \( \{b_1, b_2, \ldots, b_m\} \) then
\[
\text{proj}_V(x) = (b_1, x) b_1 + (b_2, x) b_2 + \ldots + (b_m, x) b_m
\]
In particular if \( V = \mathbb{R}^n \)
\[
x = (b_1, x) b_1 + (b_2, x) b_2 + \ldots + (b_n, x) b_n
\]

Find the orthogonal projection of \( (1,2,3) \) onto the
subspace of \( \mathbb{R}^3 \) of equation \( x+y+z=0. \)
Write \( (1,2,3) \) as a linear combination of the vectors \( v_i = (1/\sqrt{2})(1,0,1), \)
\( v_2 = (0,1,0), \) and \( v_3 = (1/\sqrt{2})(1,0,-1). \)

Consider \( V \) a subspace of \( \mathbb{R}^n. \) The orthogonal complement \( V^\perp \) of \( V \) is
the set of vectors \( x \) of \( \mathbb{R}^n \) that are orthogonal to all vectors in \( V. \)
• In other words \( V^\perp \) is the kernel of the linear transformation \( \text{proj}_V. \)
If \( V \) is a subspace of \( \mathbb{R}^n \)
• The orthogonal complement of \( V \) is a subspace of \( \mathbb{R}^n \)
• \( V \cap V^\perp = \{0\} \)
• \( \dim(V) + \dim(V^\perp) = n \)
• \( \dim(V^\perp) = V \)
• \( \dim(V^\perp) = V \)

Find the orthogonal complement \( V \) where \( V \) is the subspace
of \( \mathbb{R}^3 \) of equation \( x+y+z=0. \)

Theorem: Consider two vectors \( x \) and \( y \) in \( \mathbb{R}^n \)
• \( ||x+y||^2 = ||x||^2 + ||y||^2 \) if and only if \( x \) and \( y \) are orthogonal
(Pythagorean theorem)
• If \( V \) is a subspace of \( \mathbb{R}^n \) then
\[
||\text{proj}_V(x)|| \leq ||x||
\]
• Cauchy-Schwarz Inequality:
\[
|\langle x, y \rangle| \leq ||x|| \cdot ||y||.
\]
Consider two non-zero vectors \( x \) and \( y \) in \( \mathbb{R}^n. \) The angle \( \theta \)
between these two vectors is defined as \( \arccos\langle x, y \rangle / ||x|| \cdot ||y||. \)

Find the angle between the vectors \( x = (1,1,1) \) and \( (1,0,1). \)