Problem 1.3.5, 4ed

- Write the system below in vector form.
  - \( x + 2y = 7 \)
  - \( 3x + y = 11 \)
- Use your answer to represent the system and represent the solution geometrically.
- (not from the book) Write the system below in vector form.
  - \( x + 2y + 4z = 7 \)
  - \( 3x + y - 2z = 11 \)
- Arguing geometrically, show that this system has infinitely many solutions.

Definition

A vector \( b \) in \( \mathbb{R}^m \) is a linear combination of the vectors \( v_1, v_2, \ldots, v_n \) in \( \mathbb{R}^m \) if there exist scalars \( x_1, x_2, \ldots, x_n \) such that

\[ b = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n \]

Think of vectors as columns
(in all this slides)

Problem

Determine whether the vector \((1,4)\) is a linear combination of the vectors \((5,8)\) and \((3,5)\).

Think of vectors as columns
(in all this slides)

Review: Definition of Function

A function is a relation between two sets, the domain and the range such that each element of the domain is associated with at least one element of range.

1. \( F(x,y) = x \)
2. \( G(x,y,z) = (x,x+y) \)
3. \( H(x,y,z) = (x^2, y) \)
4. \( T(x,y) = (x,y) \)
5. \( U(x) = \sin(x) \)
6. \( V(x) = 2x + 3 \)
7. \( Z(x) = 1/x \)
8. \( W(x,y) = (y + 1, -10x, 2x + 3y) \)

Think of vectors as columns
(in all this slides)

Examples

- Consider two vectors \( a = (a_1,a_2) \) and \( x = (x_1,x_2) \).
- The dot product \( a \cdot x \) is a scalar.
- The assignment \( x \mapsto a \cdot x \) (or \( f(x) = a \cdot x \)) is a function.
- This is an example of linear function.
- What are the domain and range?
Example of function

- Consider a 2 x 2 matrix \( A \) and a vector \( x = (x_1, x_2) \).
  
- The product \( A \cdot x \) is a new vector with two entries.

- The assignment \( x \rightarrow A \cdot x \) is a function.

- This is an example of linear function.

Definition

- A function \( f \) from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) is a linear transformation if there exists an \( n \times m \) matrix \( A \) such that for each \( x \) in \( \mathbb{R}^m \), \( f(x) = A \cdot x \).

Which of the following functions are linear transformations?

1. \( F(x, y) = x \)
2. \( G(x, y, z) = (x, x + y) \)
3. \( H(x, y, z) = (x^2, y) \)
4. \( I(x, y) = (x, y) \)
5. \( U(x) = \sin(x) \)
6. \( V(x) = 2x + 3 \)
7. \( Z(x) = 1/x \)
8. \( W(x, y) = (y + 1, -10x, 2x + 3y) \)

Find the matrices associated to each of the functions which are linear transformations.

The identity matrix and transformation

- A function \( f \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) defined for each \( x \) in \( \mathbb{R}^n \) by \( f(x) = x \) is called the identity transformation.
- Is the identity a linear transformation?
- The matrix associated with the identity transformation is the identity matrix.
- Examples
- Is the linear transformation \( f(x, y, z) = (x, y) \) the identity transformation?

Definition

- The vectors of \( \mathbb{R}^n \), \( e_1 = (1, 0, 0, \ldots) \), \( e_2 = (0, 1, 0, 0, \ldots) \), \( \ldots \), \( e_n = (0, 0, \ldots, 0, 1) \) are called the standard vectors.

Example

- Find the image of the standard vectors under the linear transformation with matrix
  
\[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
0 & 4 & -1 & 2 \\
1 & 3 & 2 & 0
\end{pmatrix}
\]
Theorem

- If $T$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ then the matrix of $T$ is $[T(e_1), T(e_2), \ldots, T(e_m)]$.

Example

- Find the image of the standard vectors under the linear transformation with matrix

\[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
0 & 4 & -1 & 2 \\
1 & 3 & 2 & 0
\end{pmatrix}
\]

Problem

1. Consider a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(v) = 3v$ and $T(w) = \left[\frac{-1}{2} \right]w$ for the vectors $v = (1,2)$ and $w = (-3, 3)$. Sketch geometrically $T(x)$ for a given vector $x$.
2. Given a vector $v$ and a scalar $k$, find the relationship between $kT(v)$ and $T(kv)$.
3. Given two vectors, $v$ and $w$, find the relationship between $T(v)$, $T(w)$ and $T(v+w)$.

Theorem

A function $f$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ is linear if and only if

1. For every $x$ and $y$ in $\mathbb{R}^m$, $f(x+y) = f(x) + f(y)$.
2. For every $x$ in $\mathbb{R}^m$ and for every scalar $k$, $f(kx) = kf(x)$.

Recall: A function $f$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear transformation if there exists an $n \times m$ matrix $A$ such that for each $x$ in $\mathbb{R}^m$, $f(x) = Ax$.

Review:

- A function $g$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is the inverse of $f$ if for each $x$ in $\mathbb{R}^n$, $g(f(x)) = x$ and for each $y$ in $\mathbb{R}^m$, $f(g(y)) = y$.
- Example: If $f(x) = x + 4$ then $g(y) = x - 4$ is the inverse of $f$.
- The inverse of a function $f$ is denoted by $f^{-1}$.
- Note that not all functions have an inverse.
- Question: If $f$ has an inverse, $f^{-1}$, does $f^{-1}$ have an inverse? If so, what is the inverse of $f^{-1}$?

Study which of the following functions are linear transformations using the theorem below:

1. $F(x,y) = x$
2. $G(x,y,z) = (x, x+y)$
3. $H(x,y,z) = (x^2, y)$
4. $T(x,y) = (x,y)$
5. $U(x) = \sin(x)$
6. $V(x) = 2x + 3$
7. $Z(x) = 1/x$
8. $W(x,y) = (y+1, -10x, 2x+3y)$
Find the inverse of the linear transformations below if possible.

1. \( F(x) = -4x \)
2. \( G(x,y) = (5x+3y, 8x+5y) \)
3. \( H(x,y,z) = (y, y+z, z) \)

Invertible matrices

- If the linear transformation \( T(x) = Ax \) is invertible, then the inverse \( T^{-1} \) is also a linear transformation. (Can you prove it?)
- Thus, there exists a matrix \( B \) such that \( T^{-1}(y) = By \).
- The matrix \( B \) is the inverse of \( A \) and we write \( B = A^{-1} \).

Find the inverse of the matrix associated to the linear transformation \( G(x,y) = (5x+3y, 8x+5y) \).

The effect of a linear transformation demo

Definition

A linear transformation \( T \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) defined by \( T(x) = kx \), is called a scaling.

Example: Effect of scaling in \( \mathbb{R}^2 \) by a factor of 3.