Problem 1.1.31(4ed)  33(5ed)

□ Find all the polynomials \( f(t) \) of degree \( \leq 2 \) whose graph runs through the points \((1,3)\) and \((2,6)\), such that \( f'(1)=1 \).

Definition

A \( m \times n \) matrix is a rectangular array of \( m.n \) numbers arranged in \( m \) horizontal rows and \( n \) vertical columns

(Side note: the image is from wikipedia)

Solving linear equations by elimination

\( i^{th} \) step
1. If there is not variables in the system formed the \( i^{th} \) equation to the bottom, then the process ends.
2. If there is an equation of the form \( 0 = \text{number not zero} \), then the system has no solution (and the process is complete).
3. In the system formed the \( i^{th} \) equation to the bottom, find the first occurrence of the variable with smallest subindex, say \( x_k \).
4. If \( x_k \) does not appear in the \( i^{th} \) equation, swap the \( i^{th} \) equation with the first equation below that contains \( x_k \).
5. Divide the \( i^{th} \) equation by the coefficient of \( x_k \) in the \( i^{th} \) equation.
6. Eliminate \( x_k \) from all the other equations by subtracting to each of the other equations the appropriate multiple of the \( i^{th} \) equation. (If the coefficient of \( x_k \) in the \( j^{th} \) equation is \( c_j \), replace the \( i^{th} \) equation by \((i^{th} \text{ eq.} - c_j \text{ eq.})\).
Problem 1.2.45

Consider the system of equations below, where \( k \) is an arbitrary constant.

a. For which values of the constant \( k \) does the system have a unique solution.

b. When is there no solution?

c. When are there infinitely many solutions?

\[
\begin{align*}
  \begin{array}{ccc}
    x + 2y + 3z &= 4 \\
    x + ky + 4z &= 6 \\
    x + 2y + (k+2)z &= 6 \\
  \end{array}
\]

Decide whether each of the matrices below is in reduced row echelon form.

A matrix is in reduced row-echelon form if it satisfies the three following conditions.

- If a row has non-zero entries, then the first non-zero entry (from left to right) is 1. This entry is called the leading 1 or pivot.
- If a column contains a leading 1, then all other entries in that column are 0.
- If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

A system of equations has an “obvious” or “very easy to compute” solution when:

- The coefficient of the leading variable is 1. (The leading variable is the variable with smallest subindex in the equation)
- The leading variable in each equation does not appear in any other equation.
- The leading variables appear in natural order from top to bottom (that is from the smallest subindex to the largest).

Decide whether each of the matrices below is in reduced row echelon form.
Decide whether each of the matrices below is in reduced row echelon form.

Elementary row operations.
- Interchange the positions of two rows in the matrix.
- Multiply a row by a non-zero constant.
- Replace a row with the sum of that row and a multiple of any other row.

• Any matrix can be reduced to a matrix in reduced row echelon form by elementary row operations. (One way to do it is to “copy” the recipe or algorithm to solve a linear system we saw before.)
• If a matrix is associated to a linear system then by reducing it to a reduced row echelon form, we can find the solution of the system.

The rank of a matrix

The rank of a matrix \( A \), (denoted by \( \text{rank}(A) \)) is the number of leading 1’s in its reduced row echelon form.

Theorem: Each matrix can be transformed elementary row operations in an unique matrix in reduced row echelon form.

This implies that the definition of rank makes sense.
The rank of a matrix $A$ is the number of leading 1’s in its reduced row echelon form.

**Theorem:** A system with the same number of equations than variables has a unique solution if and only if ...

Theorem: A system of $n$ linear equations with $m$ variables is consistent if and only if either

- It has infinitely many solutions. In this case, the rank of $A$ is strictly less than $m$. OR
- It has exactly one solution. This holds if and only if all variables are leading. In this case, the rank of $A$ is $m$.

The rank matrix $A$ of a linear system of $n$ linear equations with $m$ variables is an $n \times m$ matrix, which satisfies the following statements:

- $\text{rank}(A) \leq n$
- $\text{rank}(A) \leq m$.
- If $\text{rank}(A) = n$ then the system is consistent (and $n \leq m$)
- If $\text{rank}(A) = m$ then the system has at most one solution.
- If $\text{rank}(A) < m$ then the system has either infinitely many solutions or none.

**Theorem:** A system with the same number of equations than variables has a unique solution if and only if ...

**Theorem:** A system of $n$ linear equations with $m$ variables is consistent if and only if either

- It has infinitely many solutions. In this case, the rank of $A$ is strictly less than $m$. OR
- It has exactly one solution. This holds if and only if all variables are leading. In this case, the rank of $A$ is $m$.

A linear system is consistent if it has one or more solutions. Otherwise, (that is, if it has no solutions) is inconsistent.

**Theorem:** A system of $n$ linear equations with $m$ variables is consistent if and only if either

- It has infinitely many solutions. In this case, the rank of $A$ is strictly less than $m$. OR
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**Vectors and Matrices**

- **Vectors**
  - Geometric interpretation
  - Vector spaces
  - Linear combinations
  - Dot product
- **Matrices**
  - Rank
  - Algebraic operations:
    - Sum
    - Multiplication by a scalar.
  - Product of a matrix and a vector
  - Algebraic rules.
- **Matrix form of a linear system**

A linear system is inconsistent if it has no solutions. Otherwise, it is consistent.

**Theorem:** A linear system is inconsistent if and only if the augmented as reduced row echelon form has a row of the form $[0 \ 0 \ 0 \ldots \ 0 \ 1]$.

If a linear system is consistent then either

- All the variables are leading (Thus the system has exactly one solution.)
- There is at least one no leading variable. (Thus the system has infinitely many solutions.)

**Relation between the number of solutions and the rank of the associated matrix.**

**Consequences of the relations between the number of equations and the number of unknown.**

**Linear systems summary continues**