MY NAME IS: SOLAR ID:

Problem	1	2	3	4	5	Total	Extra Credit
Score							
Total Score	20	20	20	20	20	100	20

MAT 211 - Introduction to linear algebra, Midterm 1

Oct 14th, 2009

SHOW ALL WORK TO GET FULL CREDIT; A CORRECT ANSWER WITH INCORRECT OR NO JUSTIFICATION **will not get credit**. CROSS OUT THE WORK YOU DO NOT WANT TO BE GRADED.

(1) A linear transformation $T\colon \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ is defined by

$$T(x_1, x_2, , x_3, x_4, x_5) = (x_2 + 3x_3 - x_5, x_1 + 7x_3, x_1 + 7x_3 - x_5, x_1 + 7x_3).$$

(a) (2 points) Find the matrix of T.
(b) (4 points) Find a basis for the image of T.
(c) (4 points) Find a basis for the kernel of T.
(d) (3 points) Determine the dimensions of the kernel and the image of T.
(e) (3 points) Determine the rank of the matrix of T.
(f) (4 points) Find vectors that span the image of T but do not form a basis.

a. $T = \begin{bmatrix} 0 & 1 & 3 & 0 & -1 \\ 1 & 0 & 7 & 0 & 0 \\ 1 & 0 & 7 & 0 & -1 \\ 1 & 0 & 7 & 0 & 0 \end{bmatrix}$

b $\mathrm{Im}(T) = \underbrace{X_1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{V_1} + \underbrace{X_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{V_2} + \underbrace{X_3 \begin{bmatrix} 3 \\ 7 \\ 7 \\ 7 \end{bmatrix}}_{V_3} + \underbrace{X_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{V_4} + \underbrace{X_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{V_5}$

V_1, V_2, V_5 are linearly independant, $V_3 = 7V_1 + 3V_2$. $V_4 = 0V_1 + 0V_2 + 0V_5$

So A basis of $\mathrm{im}(T)$ is $\{V_1, V_2, V_5\}$ (V_3 and V_4 are redundant)

c. Solve the equation $T\vec{x} = 0$

$$\begin{bmatrix} 0 & 1 & 3 & 0 & -1 \\ 1 & 0 & 7 & 0 & 0 \\ 1 & 0 & 7 & 0 & -1 \\ 1 & 0 & 7 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 7 & 0 & 0 \\ 1 & 0 & 7 & 0 & 0 \\ 1 & 0 & 7 & 0 & -1 \\ 0 & 1 & 3 & 0 & -1 \end{bmatrix} \end{array} \begin{array}{l} \\ -\mathrm{I} \\ -\mathrm{I} \\ \\ \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & 0 & -1 \end{bmatrix} \begin{array}{l} \\ \\ \\ -\mathrm{III} \end{array} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \times(-1) \\ \text{Swap II \& IV} \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 7 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the solution would be $\vec{x} = t\begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $t, s \in R$

$\begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis of the kernel of T

d. the ~~two~~ basis of $im(T)$ consists of three independant vectors, so $Dim(im(T)) = 3$. Similarly, $Dim(ker(T)) = 2$.

e. $T = \begin{bmatrix} 0 & 1 & 3 & 0 & -1 \\ 1 & 0 & 7 & 0 & 0 \\ 1 & 0 & 7 & 0 & -1 \\ 1 & 0 & 7 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ from c, so $rank(T) = 3$

f. $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(2) Consider the system

$$\begin{cases} x_1 + x_2 + x_3 &= 2 \\ x_1 - x_2 + x_4 &= 5 \\ x_2 + x_4 &= -1 \\ -x_1 + x_5 &= 7 \end{cases}$$

(a) (15 points) Solve the system by Gauss-Jordan elimination.

(b) (5 points) Check that your answer is correct.

a.
$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 2 \\ 1 & -1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 7 \end{pmatrix} \to \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -7 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 2 \\ 1 & -1 & 0 & 1 & 0 & 5 \end{pmatrix} \end{array} \begin{array}{l} (-1)\mathrm{IV} \\ \mathrm{III} \\ \mathrm{I} \\ \mathrm{II} \end{array} \to \begin{array}{l} \\ \\ -\mathrm{I}-\mathrm{II} \\ -\mathrm{I}+\mathrm{II} \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -7 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 10 \\ 0 & 0 & 0 & 2 & 1 & 11 \end{pmatrix} \begin{array}{l} \\ \\ \\ \times\frac{1}{2} \end{array} \to \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -7 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 10 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{11}{2} \end{pmatrix} \begin{array}{l} \\ -\mathrm{IV} \\ +\mathrm{IV} \\ \\ \end{array} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -7 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{13}{2} \\ 0 & 0 & 1 & 0 & \frac{3}{2} & \frac{21}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{11}{2} \end{pmatrix}$$

So the solution is $\vec{x} = t\begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} + \begin{pmatrix} -7 \\ -\frac{13}{2} \\ \frac{31}{2} \\ \frac{11}{2} \\ 0 \end{pmatrix}$

b. Check. $(t-7) + \cancel{\frac{t}{2} - \frac{t}{8}}\left(\frac{t}{2} - \frac{13}{2}\right) + \left(-\frac{3t}{2} + \frac{31}{2}\right) = \left(t + \frac{t}{2} - \frac{3t}{2}\right) + \left(-7 - \frac{13}{2} + \frac{31}{2}\right) = 2.$

$(t-7) - \left(\frac{t}{2} - \frac{13}{2}\right) + \left(-\frac{t}{2} + \frac{11}{2}\right) = \left(t - \frac{t}{2} - \frac{t}{2}\right) + \left(-7 + \frac{13}{2} + \frac{11}{2}\right) = 5$

$\cancel{-\frac{3}{2}t + \frac{31}{2}}\ \left(\frac{1}{2}t - \frac{13}{2}\right) + \left(-\frac{t}{2} + \frac{11}{2}\right) = -1$

$-(t-7) + t = 7$

So the solution in a is correct.

(3) Consider the line L in R^3 of multiples of $(2, 2, 1)$. Denote by $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ the orthogonal projection onto L.

(a) (4 points) Find the matrix of T.

(b) (4 points) Describe geometrically the kernel and image of T.

(c) (4 points) Denote by v the vector $(1, 2, 0)$. Find $T(v)$.

(d) (4 points) Is T a linear transformation?

(e) (4 points) Determine whether T is invertible and if it is, find T^{-1}.

a. $$T\vec{x} = \left(\frac{\langle \vec{x}, (2,2,1)\rangle}{|(2,2,1)|}\right) \cdot \frac{(2,2,1)}{|(2,2,1)|}$$

$$= \left(\tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 + \tfrac{1}{3}x_3\right) \cdot \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right)$$

$$= \left(\tfrac{4}{9}x_1 + \tfrac{4}{9}x_2 + \tfrac{2}{9}x_3 \quad \tfrac{4}{9}x_1 + \tfrac{4}{9}x_2 + \tfrac{2}{9}x_3,\ \tfrac{2}{9}x_1 + \tfrac{2}{9}x_2 + \tfrac{1}{9}x_3\right)$$

so $$T = \begin{bmatrix} \frac{4}{9} & \frac{4}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{4}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{bmatrix}$$

b. ker T is a plane passing through $(0,0,0)$ and ~~par~~ orthogonal to the line L of multiples of $(2,2,1)$

c. $$T(v) = \begin{bmatrix} \frac{4}{9} & \frac{4}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{4}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{4}{3} \\ \frac{2}{3} \end{bmatrix}$$

d. Yes. because $T(x) = Ax$ where A is a matrix.

e. No $\mathrm{Dim}(\ker T) = 2 \neq 0$.

(4) (a) (15 points) Find a 2×2 matrix A such that $A^3 = I_2$ and all entries of A are not zero. (Recall that I_2 is the 2×2 identity matrix). (Hint: Think of a geometric transformation)

(b) (5 points) Compute A^4.

a. Let A be $\begin{bmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix} = \left(\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \right)$

Then $A^3 = I_2$

b. $A^4 = (A^3) \cdot A = I_2 \cdot A = A = \left(\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \right)$

(5) Consider the matrix

$$A = \begin{pmatrix} (k-1) & 0 & 0 \\ 0 & (k^2-1) & 0 \\ 0 & 0 & (k+1) \end{pmatrix}$$

(a) (11 points) Find all the values of k for which A is invertible and compute the inverse of A for those values.

(b) (3 points) Are there any values of k for which A has rank 1? If so, indicate them.

(c) (3 points) Are there any values of k for which A has rank 2? If so, indicate them.

(d) (3 points) Are there any values of k for which A has rank 3? If so, indicate them.

a If A is invertable. $\det(A) = (k-1)(k^2-1)(k+1) \neq 0$, then $k \neq \pm 1$. In that case $A^{-1} = \begin{bmatrix} \frac{1}{k-1} & 0 & 0 \\ 0 & \frac{1}{k^2-1} & 0 \\ 0 & 0 & \frac{1}{k+1} \end{bmatrix}$

b. c. If ran $A = 1$ or 2, then $\det A = 0$, so $k = 1$ or $k = -1$

If $k = 1$, $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\operatorname{rank}(A) = 1$.

If $k = -1$, $A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\operatorname{rank}(A) = 1$

Therefore, if $k = \pm 1$, $\operatorname{rank}(A) = 1$. No value of k makes $\operatorname{rank}(A) = 2$

d. when $k \neq \pm 1$

(6) $\boxed{\text{Extra credit}}$ Show that the kernels of the matrices A and B are different.

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -1 & 0 & 7 & 0 \\ 0 & 1 & -1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Apply Algorithm

$$\begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow V_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 7 \\ 3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \qquad V_2 = \begin{pmatrix} -3 \\ -7 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

$\{V_1, V_2\}$ is a basis of $\ker A$

Now we prove that $V_2 \notin \ker B$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 7 & 0 \\ 0 & 1 & -1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -7 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3+7 \\ \vdots \end{pmatrix} = \begin{pmatrix} 4 \\ \vdots \end{pmatrix}$$ not in $\ker B$