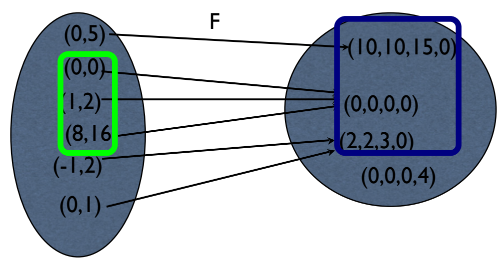
MAT211

Image and Kernel of a Linear Transformation

Domain: (Subset of)R^2 Target: (Subset of)R^4



Definition

The image of a function f : X -> Y is the subset of elements y of Y which are of the form f(x) for some x in X.

The image of a function f is denoted by im(f).

Example

- Describe the image of the linear transformation f(x,y)=(3x+6y,x+2y)

Definition

Let $v_1, v_2,.., v_m$ be in R^n. The span of $v_1, v_2,.., v_m$ is the set of all linear combinations

$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m$

We denote it by span($v_1, v_2,.., v_m$)

Question

Consider two vectors v and w in R^n. Describe geometrically span(v) and span(v,w).

Theorem

The image of a linear transformation T(x)=Ax is the span of the columns of A.

(Compare this theorem with the previous Example)

Theorem: Properties of the Image

Consider a linear transformation T from R^m to R^n

The zero vector is in the image.

If x and y are in the image, then x+y is in the image.

If x is in the image and k is an scalar, then k.x is in the image.

EXAMPLE

Find the image of the linear transformation of matrix
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

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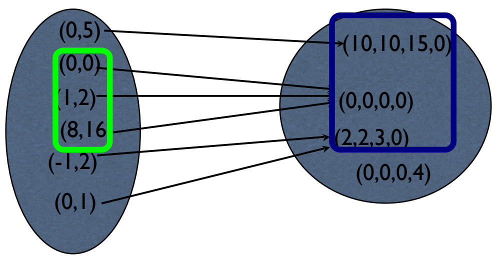

Definition

The kernel of a function f : X -> Y is the subset of elements x of X for such that f(x)=0.

The kernel of a function f is denoted by ker(f)

Theorem: Properties of the Kernel

Consider a linear transformation T from R^m to R^n

The zero vector is in the kernel.

If x and y are in the kernel, then x+y is in the kernel.

If x is in the kernel and k is an scalar, then k.x is in the kernel.

Example

For each matrix A,

- Find vectors that span the image of A.
- Find vectors that span the kernel of A.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

In all cases, give as few vectors as possible.

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix}$$

Example

- Describe geometrically the kernel and image of the orthogonal projection onto the line L of equation y=2x+1.

An n x n matrix A is invertible if and only if one of the following holds

- The linear system A.x=b has a unique solution.
- rref(A)=Id
- rank(A)=n
- im(A)=R^n
- ker(A)={0}

Definition

- A subset W of R^n is a (linear) subspace of R^n if satisfies the following

1. W contains the zero vector.
2. If v and w are in W then v+w are in W.
3. If v is in W and k is any scalar then k.v is in W.

EXAMPLE: Are the following sets subspaces?

1. A line L in R^2
2. The union of two lines in R^2
3. A plane in R^3
4. {0}
5. The kernel and image of a linear transformation.

Definition

Consider vectors v_1, v_2, .. v_m in R^n.

- A vector v_i is redundant if v_i is a linear combination of v_1, v_2, .. v_{i-1}.
- The vectors v_1, v_2, .. v_m are linearly independent if none of them is redundant.