Definition

The image of a function \( f : X \rightarrow Y \) is the subset of elements \( y \) of \( Y \) which are of the form \( f(x) \) for some \( x \) in \( X \).

The image of a function \( f \) is denoted by \( \text{im}(f) \).

Example

- Describe the image of the linear transformation \( f(x,y) = (3x+6y, x+2y) \).

Definition

Let \( v_1, v_2, \ldots, v_m \) be in \( \mathbb{R}^n \). The span of \( v_1, v_2, \ldots, v_m \) is the set of all linear combinations \( c_1v_1 + c_2v_2 + \ldots + c_mv_m \).

We denote it by \( \text{span}(v_1, v_2, \ldots, v_m) \).

Question

Consider two vectors \( v \) and \( w \) in \( \mathbb{R}^n \). Describe geometrically \( \text{span}(v) \) and \( \text{span}(v,w) \).
The image of a linear transformation $T(x)=Ax$ is the span of the columns of $A$.

(Compare this theorem with the previous Example)

**Theorem: Properties of the Image**

Consider a linear transformation $T$ from $\mathbb{R}^m$ to $\mathbb{R}^n$. The zero vector is in the image. If $x$ and $y$ are in the image, then $x+y$ is in the image. If $x$ is in the image and $k$ is a scalar, then $kx$ is in the image.

**EXAMPLE**

Find the image of the linear transformation of matrix

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 3 & 6 \\
0 & 0 & 3
\end{pmatrix}
$$

**Definition**

The kernel of a function $f : X \rightarrow Y$ is the subset of elements $x$ of $X$ for which $f(x)=0$. The kernel of a function $f$ is denoted by $\text{ker}(f)$.

**Theorem: Properties of the Kernel**

Consider a linear transformation $T$ from $\mathbb{R}^m$ to $\mathbb{R}^n$. The zero vector is in the kernel. If $x$ and $y$ are in the kernel, then $x+y$ is in the kernel. If $x$ is in the kernel and $k$ is a scalar, then $kx$ is in the kernel.
Example

For each matrix $A$,

- Find vectors that span the image of $A$.
- Find vectors that span the kernel of $A$.

In all cases, give as few vectors as possible.

\[
\begin{pmatrix}
1 & 0 & 1 \\
1 & 2 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
\end{pmatrix}
\]

Example

- Describe geometrically the kernel and image of the orthogonal projection onto the line $L$ of equation $y=2x+1$.

Definition

An $n \times n$ matrix $A$ is invertible if and only if one of the following holds

- The linear system $A \cdot x = b$ has a unique solution.
- $\text{rref}(A) = \text{Id}$
- $\text{rank}(A) = n$
- $\text{im}(A) = \mathbb{R}^n$
- $\text{ker}(A) = \{0\}$

Definition

A subset $W$ of $\mathbb{R}^n$ is a (linear) subspace of $\mathbb{R}^n$ if it satisfies the following

1. $W$ contains the zero vector.
2. If $v$ and $w$ are in $W$ then $v + w$ are in $W$.
3. If $v$ is in $W$ and $k$ is any scalar then $k \cdot v$ is in $W$.

Example: Are the following sets subspaces?

1. A line $L$ in $\mathbb{R}^2$
2. The union of two lines in $\mathbb{R}^2$
3. A plane in $\mathbb{R}^3$
4. $\{0\}$
5. The kernel and image of a linear transformation.

Definition

Consider vectors $v_1, v_2, \ldots, v_m$ in $\mathbb{R}^n$.

- A vector $v$, is redundant if $v$, is a linear combination of $v_1, v_2, \ldots, v_{i-1}$.
- The vectors $v_1, v_2, \ldots, v_m$ are linearly independent if none of them is redundant.