**MAT211 Lecture 6-7**

Linear Transformations in Geometry

Matrix Products and Inverses

L is a line through the origin, u is vector in L, u.u=1

<table>
<thead>
<tr>
<th>Name</th>
<th>Formula</th>
<th>Notation</th>
<th>Matrix</th>
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<tbody>
<tr>
<td>Scaling (dilation if k&gt;1; contraction if k&lt;1)</td>
<td>k.x</td>
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<tr>
<td>Orthogonal projection onto a line L in R²</td>
<td>(u.x)u</td>
<td>proj.(x)</td>
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<tr>
<td>Reflection about a line L in R²</td>
<td>2(u.x)u-x</td>
<td>ref,(x)</td>
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<tr>
<td>Rotation through a fixed angle θ</td>
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V is a plane through the origin, orthogonal to L

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<tr>
<td>Reflection about a plane V in R³</td>
<td>x-2(u.x)u</td>
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<td>Orthogonal projection onto a line L in R³</td>
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Exercise: Find the matrices

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**Example**

*Interpret geometrically the following linear transformation.*

\[ T(x, y) = (x - \sqrt{3}y, \sqrt{3}x + y); \]

---

**Example (2.2-7)**

L be the line in R that consists of all scalar multiples of (2,1,2). Find the reflection of the vector (1,1,1) about the line L.

---

**Example (2.2- 8 and 9)**

\[
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}
\]

*Interpret geometrically the following linear transformations.*

\[
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\]
Consider the transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) defined by 
\[
 f(x_1, x_2, x_3) = x_1 \left( \begin{array}{c} 10 \\ 2 \\ 1 \end{array} \right) + x_3 \left( \begin{array}{c} 1 \\ -2 \end{array} \right)
\]

Determine whether this transformation is linear.

Determine whether this transformation is invertible.

### Overview

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<tr>
<th>Function</th>
<th>Linear Transformations</th>
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<td>( f(x) = Ax )</td>
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<td>Composition of linear transformations</td>
<td>Product of matrices</td>
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<td>Invertible linear transformations</td>
<td>Invertible matrices</td>
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<td>Inverse of a linear transformation</td>
<td>( A^{-1} )</td>
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Let \( A \) be an \( m \times n \) matrix and \( B \) a \( p \times q \) matrix. The product \( AB \) is defined if and only if \( n = q \).

Let \( A \) be an \( m \times p \) matrix and \( B \) a \( p \times q \) matrix. The product \( AB \) is the matrix of the linear transformation \( T(x) = A(Bx) \). The product \( AB \) is an \( m \times q \) transformation.

Let \( B \) be an \( n \times p \) matrix and let \( A \) be a \( p \times m \) matrix with columns \( v_1, v_2, \ldots, v_m \). Then the product \( BA \) is equal to the matrix

\[
\begin{pmatrix}
Bv_1 & Bv_2 & \cdots & Bv_m
\end{pmatrix}
\]

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\[
\begin{pmatrix}
b_{11}a_{1j} + b_{12}a_{2j} + \cdots + b_{1p}a_{pj}
\end{pmatrix}
\]

Compute the products \( AB \) and \( BA \).

\[
A = \begin{pmatrix} 3 & 2 \\ 1 & 10 \end{pmatrix}
\]

\[
B = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}
\]
The product of matrices is not commutative. This is, in general $A \cdot B \neq B \cdot A$.

The product of matrices is associative: $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

The distributive law holds for matrices $A \cdot (B + C) = A \cdot B + A \cdot C$.

Find the products $A \cdot B$ and $B \cdot A$.

The identity matrix $I_n$ of size $n$ is the matrix

$$
I_n = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
$$

If $A$ is an $n \times m$ matrix then $A \cdot I_m = I_n \cdot A = A$.

Multiply the matrices, whenever is possible

$$
A = \begin{pmatrix}
1 & 4 & 0 \\
0 & 3 & -1
\end{pmatrix}, \quad B = \begin{pmatrix}
3 & -1 \\
5 & 1
\end{pmatrix}
$$

Recall

A function $f: X \to Y$ is invertible if for every element $y$ in $Y$, the equation $f(x) = y$ has a unique solution.

The inverse of an invertible function $f$ is denoted by $f^{-1}$.
**Definition**

- A matrix $A$ is invertible if it is square (that is of size $n \times n$) and the associated linear transformation is invertible.
- If a linear transformation $T(x) = Ax$ is invertible, the associated matrix of the inverse $T^{-1}$ is denoted by $A^{-1}$.

**Prove the following facts**

- The $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - cb \neq 0$.
- Find the inverse of $A$

**EXAMPLE (2.4-60)**

- Show that the following matrix is invertible.
- Interpret geometrically the associated linear transformation.

$$\begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}$$

**Find the inverse**

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

**Theorem**

- To find the inverse of a matrix $A$ of size $n \times n$, compute the reduced row echelon form of the matrix $(A \mid I_n)$
- If $\text{rref}(A \mid I_n) = (I_n \mid B)$ for some matrix $B$, then $A$ is invertible and $B = A^{-1}$
- Otherwise, $A$ is not invertible.

**Theorem**

- If $A$ is an invertible matrix of size $n \times n$ then $A^{-1}A = I_n$ and $AA^{-1} = I_n$
- If $A$ and $B$ are invertible matrices then the matrix $A \cdot B$ is invertible and $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$
- If $A$ and $B$ are two matrices of size $n \times n$ such that $BA = I_n$, then $A$ and $B$ are invertible, $A = B^{-1}$ and $B = A^{-1}$. 
EXAMPLE (2.4=29)

For which values of the constant \( k \) is the following matrix invertible?

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & k \\
1 & 4 & k^2 \\
\end{pmatrix}
\]