MAT211 Lecture 6-7

Linear Transformations in Geometry
Matrix Products and Inverses

L is a line through the origin, u is vector in L, u.u=1

Name	Formula	Notation	Matrix
Scaling (dilation if k>1; contraction if k<1)	k.x k scalar		
Orthogonal projection onto a line L in R^2	(u.x)u	$proj_L(x)$	
Reflection about a line L in R^2	2(u.x)u-x	$ref_L(x)$	
Rotation through a fixed angle θ			$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

L is a line through the origin, u is vector in L, u.u=1
V is a plane through the origin, orthogonal to L

Name	Formula	Notation	Matrix
Reflection about a plane V in R^3	x-2(u.x)u	$ref_V(x)$	
Orthogonal projection onto a line L in R^3	(u.x)u	$proj_L(x)$	
Orthogonal projection onto a plane V in R^3	x - (u.x)u	$proj_V(x)$	
Reflection about a line L in R^3	2(u.x)u-x	$ref_L(x)$	

Exercise: Find the matrices

Example

- Interpret geometrically the following linear transformation.

$$T(x,y) = (x - \sqrt{3}y, \sqrt{3}x + y)$$

Example (2.2-7)

L be the line in R that consists of all scalar multiples of (2,1,2). Find the reflection of the vector (1,1,1) about the line L.

Example (2.2- 8 and 9)

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

- Interpret geometrically the following linear transformations.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

- Consider the transformation from R^3 to R^2 defined by $f(x_1,x_2,x_3)=$ $x_1 \begin{pmatrix} 10 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
- Determine whether this transformation is linear
- Determine whether this transformation is invertible.

Overview

Function	
Linear Transformations	f(x)=A.x
Composition of linear transformations	Product of matrices
Invertible linear transformations	Invertible matrices
Inverse of a linear transformation	A^{-1}

- Let A be an m x n matrix and B a p x q matrix. The product A.B is defined if and only if n=q.
- Let A be an m x p matrix and B a p x q matrix. The product A.B is the matrix of the linear transformation T(x)=A(Bx). The product A.B is an m x q transformation.

- Let B be an n x p matrix and let A be a p x m matrix with columns v_1, v_2,..,v_m. Then the product B.A is equal to the matrix

$$\begin{pmatrix} | & | & \cdots & | \\ Bv_1 & Bv_2 & \cdots & Bv_m \\ | & | & \cdots & | \end{pmatrix}$$

- Let B be an n x p matrix and let A be a p x m matrix with columns v_1, v_2,..,v_m. Then the product i j entry (row i, column j) of the product B.A is given by

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{ip}a_{pj}$$

- Compute the products A.B and B.A. $A = \begin{pmatrix} 3 & 2 \\ 1 & 10 \end{pmatrix}$

$$B = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

- The product of matrices is not commutative. This is, in general A.B ≠ B.A.
- The product of matrices is associative: (A.B).C=A.(B.C).
- The distributive law holds for matrices A.(B+C)=A.B+A.C

- Find the products A.B and B.A

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

- The identity matrix I_n of size n is the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- If A is an n x m matrix then $A.I_m = I_n.A = A$

Multiply the matrices, whenever is possible

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 3 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -1 \\ 5 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 4/5 \\ 3 & -1 \\ 5 & 1 \end{pmatrix}$$

Recall

- A function f:X -> Y is invertible if for every element y in Y, the equation f(x)=y has a unique solution.
- The inverse of an invertible function f is denoted by f^{-1}

Definition

- A matrix A is invertible if it is square (that is of size n x n) and the associated linear transformation is invertible.
- If a linear transformation T(x)=Ax is invertible, the associated matrix of the inverse T^{-1} is denoted by A^{-1}

- Prove the following facts
  - The 2 x 2 matrix A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if ad-cb≠0.
  - Find the inverse of A.

EXAMPLE(2.4-60)

- Show that the following matrix is invertible.
- Interpret geometrically the associated linear transformation.

$$\begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}$$

Find the inverse

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

Theorem

- To find the inverse of a matrix A of size n x n, compute the reduced row echelon form of the matrix $(A \mid I_n)$
  - If $rref(A \mid I_n) = (I_n \mid B)$ for some matrix B, then A is invertible and $B=A^{-1}$
  - Otherwise, A is not invertible.

Theorem

- If A is an invertible matrix of size n x n then $A^{-1}.A=I_n$ and $A.A^{-1}=I_n$
- If A and B are invertible matrix then the matrix A.B is invertible and $(A.B)^{-1}=B^{-1}A^{-1}$
- If A and B are two matrices of size n x n such that $BA=I_n$ then A and B are invertible, $A=B^{-1}$ and $B=A^{-1}$.

EXAMPLE (2.4=29)

- For which values of the constant k is the following matrix invertible?

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$$