MAT211 Lecture 18

Eigenvalues and eigenvectors

Consider the projection P on R^2 onto the x-axis.
Find all vectors v such that P(v) is parallel to v.

Consider the reflection on R^2 with respect to the x-axis.
Find all vectors v such that R(v) is parallel to v.

Definition

Consider an n x n matrix A.

- A vector v in Rn is an *eigenvector* if Av is a multiple of v, that is, if there exists a scalar k such that Av=kv.
- A scalar k such that Av=kv for some vector v is an *eigenvalue*.

Example

- Consider an orthogonal projection onto a plane P on R^3. Find all the eigenvalues and eigenvectors.
- Consider reflection with respect to a plane P on R^3. Find all the eigenvalues and eigenvectors.
- Consider reflection with respect to a line L on R^3. Find all the eigenvalues and eigenvectors.

Example

- Consider the matrix of a rotation of angle π/3 in R^2. Find all the eigenvalues and eigenvectors.
- What are the eigenvalues and eigenvectors of any rotation?

Example 7.2-29

- Consider an n x n matrix A such that the sum of the entries of each row is 1. Show that the vector (1,1,...1) is an eigenvector.
- What is the corresponding eigenvalue?

Consider the matrix A

$$\begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix}$$

Find all eigenvalues and eigenvectors

Definition

Consider an n x n matrix A. The polynomial

$P(\lambda)= \det(A-\lambda I_n)$

called the *characteristic polynomial* of A.

Theorem

Consider an n x n matrix A. A scalar λ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial of A, that is if and only if

$\det(A-\lambda I_n)=0$.

Definition:

$\boxed{\det(A-\lambda I_n)=0}$

is called the *characteristic equation* ofA.

Example

Find the eigenvalues

$$\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

Theorem

- If λ is an eigenvalue of an n x n matrix A, then the associated eigenvectors form the kernel of the transformation $(A-\lambda I_n)$.
- In other words, v is an eigenvector with eigenvalue λ if an only if

$$\boxed{(A-\lambda I_n).v=0}$$

Example

Find the eigenvalues and associated eigenvectors.

$$\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

Definition

- Consider an eigenvalue λ of an n x n matrix A. The kernel of the matrix $(A-\lambda I_n)$ is called the *eigenspace associated with* λ and denoted by E_λ. In symbols,

$$\boxed{E_\lambda = \ker(A-\lambda I_n)=\{v \text{ in } R^n : Av= \lambda v\}}$$

Review: Consider n x n matrix A.

Eigenvalues: solutions λ in R of $\det(A-\lambda I_n)=0$

Eigenvectors: v in R^n solution of $(A-\lambda I_n).v=0$

Eigenspace subspace of R^n
$E_\lambda = \ker(A-\lambda I_n)$

Example 7.1-41

- Find a basis of the linear space V of all 2 x2 matrices A for which (0,1) is an eigenvector
- Find a basis of the linear space V of all 2 x2 matrices A for both (1,1) and (1,2) are eigenvectors.
- IN both cases, determine the dimension of V.

Definition:

- If A is a square matrix, the sum of the diagonal entries of A is called the *trace* of A, and denoted by tr(A).

Example

- Find the trace of the identity m

Example

Find the trace of the following matrices.

$$\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{vmatrix}$$

Theorem:

- If A is an n x n matrix then the characteristic polynomial of A has the form

  $(-\lambda)^n + tr(A)(-\lambda)^{n-1} + ... + det(A)$.

  In particular, if n=2 then the characteristic polynomial of A is

  $\lambda^2 - tr(A)\lambda + det(A)$.

Example. 7.2-15

- Consider the matrix A, where k is an arbitrary constant. For which values of k does A have two distinct real eigenvalues? When is there no real eigenvalue?
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$$\begin{vmatrix} 1 & k \\ 1 & 1 \end{vmatrix}$$

Definition

An eigenvalue λ_0 of an n x n matrix A has *algebraic multiplicity k* if it is a root of multiplicity k of the characteristic polynomial of A. In symbols, if

$det(A-\lambda I_n)=(\lambda_0-\lambda)^k g(\lambda)$

for some polynomial $g(\lambda)$ such that $g(\lambda_0) \neq 0$

Example

Find the eigenvalues with their multiplicity.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \qquad \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$