Consider the projection $P$ on $\mathbb{R}^2$ onto the x-axis. Find all vectors $v$ such that $P(v)$ is parallel to $v$.

Consider the reflection on $\mathbb{R}^2$ with respect to the x-axis. Find all vectors $v$ such that $R(v)$ is parallel to $v$.

**Definition**

Consider an $n \times n$ matrix $A$.
- A vector $v$ in $\mathbb{R}^n$ is an eigenvector if $Av$ is a multiple of $v$, that is, if there exists a scalar $k$ such that $Av = kv$.
- A scalar $k$ such that $Av = kv$ for some vector $v$ is an eigenvalue.

**Example**

- Consider an orthogonal projection onto a plane $P$ on $\mathbb{R}^3$. Find all the eigenvalues and eigenvectors.
- Consider reflection with respect to a plane $P$ on $\mathbb{R}^3$. Find all the eigenvalues and eigenvectors.
- Consider reflection with respect to a line $L$ on $\mathbb{R}^3$. Find all the eigenvalues and eigenvectors.

**Example**

- Consider the matrix of a rotation of angle $\pi/3$ in $\mathbb{R}^2$. Find all the eigenvalues and eigenvectors.
- What are the eigenvalues and eigenvectors of any rotation?
Example 7.2-29

- Consider an $n \times n$ matrix $A$ such that the sum of the entries of each row is 1. Show that the vector $(1,1,...,1)$ is an eigenvector.
- What is the corresponding eigenvalue?

Consider the matrix $A$

\[
\begin{pmatrix}
1 & 1 \\
-2 & 4
\end{pmatrix}
\]

Find all eigenvalues and eigenvectors

Definition

Consider an $n \times n$ matrix $A$. The polynomial

\[ P(\lambda) = \det(A-\lambda I) \]

called the characteristic polynomial of $A$. 

Theorem

Consider an $n \times n$ matrix $A$. A scalar $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is a root of the characteristic polynomial of $A$, that is if and only if $\det(A-\lambda I)=0$.

Example

Find the eigenvalues

\[
\begin{pmatrix}
0 & -1 & 1 & 2 & -1 \\
-1 & 0 & 1 & 0 & 1 \\
0 & 1 & 4 & -4 & 5 \\
-1 & 0 & & & 
\end{pmatrix}
\]
Theorem
- If $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$, then the associated eigenvectors form the kernel of the transformation $(A-\lambda I_n)$.
- In other words, $v$ is an eigenvector with eigenvalue $\lambda$ if and only if $(A-\lambda I_n)v=0$.

Example
Find the eigenvalues and associated eigenvectors.

$$
\begin{pmatrix}
0 & -1 \\
-1 & 0 \\
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 & -1 \\
1 & 0 & 1 \\
4 & -4 & 5
\end{pmatrix}
$$

Definition
- Consider an eigenvalue $\lambda$ of an $n \times n$ matrix $A$. The kernel of the matrix $(A-\lambda I_n)$ is called the eigenspace associated with $\lambda$ and denoted by $E_\lambda$. In symbols,

\[E_\lambda = \ker(A-\lambda I_n) = \{v \in \mathbb{R}^n : Av = \lambda v\}\]

Review: Consider $n \times n$ matrix $A$.
- Eigenvalues: solutions $\lambda$ in $\mathbb{R}$ of $\det(A-\lambda I_n)=0$
- Eigenvectors: $v$ in $\mathbb{R}^n$ solution of $(A-\lambda I_n)v=0$
- Eigenspace subspace of $\mathbb{R}^n$ $E_\lambda = \ker(A-\lambda I_n)$

Example 7.1-41
- Find a basis of the linear space $V$ of all $2 \times 2$ matrices $A$ for which $(0,1)$ is an eigenvector.
- Find a basis of the linear space $V$ of all $2 \times 2$ matrices $A$ for both $(1,1)$ and $(1,2)$ are eigenvectors.
- In both cases, determine the dimension of $V$.

Definition:
- If $A$ is a square matrix, the sum of the diagonal entries of $A$ is called the trace of $A$, and denoted by $\text{tr}(A)$. 

Example

• Find the trace of the identity matrix

\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\]

Find the trace of the following matrices.

\[
\begin{bmatrix}
0 & -1 & 1 & 2 & -1 \\
-1 & 0 & 1 & 0 & 1 \\
4 & -4 & 5
\end{bmatrix}
\]

Example

Find the trace of the following matrices.

\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & -1 \\
1 & 0 & 1 \\
4 & -4 & 5
\end{bmatrix}
\]

Theorem:

• If \( A \) is an \( n \times n \) matrix then the characteristic polynomial of \( A \) has the form

\[
(-\lambda)^n + \text{tr}(A)(-\lambda)^{n-1} + \ldots + \text{det}(A).
\]

In particular, if \( n=2 \) then the characteristic polynomial of \( A \) is

\[
\lambda^2 - \text{tr}(A)\lambda + \text{det}(A).
\]

Example. 7.2-15

• Consider the matrix \( A \), where \( k \) is an arbitrary constant. For which values of \( k \) does \( A \) have two distinct real eigenvalues? When is there no real eigenvalue?

\[
\begin{bmatrix}
1 & k \\
1 & 1
\end{bmatrix}
\]

Definition

An eigenvalue \( \lambda_0 \) of an \( n \times n \) matrix \( A \) has algebraic multiplicity \( k \) if it is a root of multiplicity \( k \) of the characteristic polynomial of \( A \). In symbols, if

\[
\text{det}(A-\lambda I_n)=(\lambda_0-\lambda)^k g(\lambda)
\]

for some polynomial \( g(\lambda) \) such that \( g(\lambda_0)\neq0 \)

Example

Find the eigenvalues with their multiplicity.

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]