MAT211 Lecture 17

Determinants.

Recall that a 2 x 2 matrix A

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is invertible

If and only if a.d-b.c ≠ 0.

The number a.d-b.c is the the determinant of A.

Definition

A *pattern* in an n x n matrix A is a choice of entries of A, such that there is only one entry in each row and only one in each column.

Example

- Find all possible patterns in 2x2 matrices
- Find all possible patterns in 3x3 matrices.
- How many patterns are there in 4x4 matrices? And in n x n matrices?

Definition

The entries of a pattern P of a matrix A are inverted if one of them is located right and above the other in A.

The signature of a pattern P of a matrix A, denoted by sgn(P) is

$(-1)^{(\text{number of pairs of inverted entries in P})}$

EXAMPLE

- Find the signature of the pattern of a 3 x 3 matrix A,

  a_{31}, a_{22}, a_{13}

Definition

The determinant a square matrix A, denoted by det A is

$\sum$sgn(P).product(elements in P)

where the sum is taken over all patterns P.

EXAMPLE

Using the definition,compute the determinant of 2x2 and 3x3 matrices.

Sarus rule.

Example

Compute the determinant of

$$\begin{vmatrix} 1 & 10 \\ 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 1 & 10 & 2 \\ -1 & 0 & 4 \\ 5 & 6 & 6 \end{vmatrix}$$

Theorem: Consider square matrices A and B.

- If A is upper triangular then det(A) is the product of the diagonal entries of A.
- det(A)=det(A^t)
- det(A.B) = det(A).det(B)
- If A and B are similar, det(A)=det(B).
- If A is invertible det(A^{-1})=1/det(A).

Example

- Compute the determinant of the inverse of

$$\begin{vmatrix} 1 & 10 & 0 & 3 & 1 \\ 0 & -1/6 & 0 & 5 & 45 \\ 0 & 0 & 6 & 5 & 99 \\ 0 & 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

Theorem: Elementary row operations and determinants

- If B is obtained from A by multiplying a r of A by a scalar k then

  det(B) = k.det(A)
- If B is obtained from A by swaping two rows then det(B)=-det(A)
- If B is obtained from A by adding a multiple of a row to another row then det(B)=det(A).

Definition

- Consider a linear space V and a linear transformation T from V to V. The determinant of T is the determinant of the matrix of T with respect to any basis of V.

13

EXAMPLE

- Find the determinant of the transformations
  - From P_2 to P_2, T(f)=2f- 3f'
  - From U^{2x2} to U^{2x2} T(M)=AM, where A is

$$\begin{vmatrix} -1 & -10 \\ 0 & -100 \end{vmatrix}$$

Example:

Of an 4 x 4 matrix A with columns v_1, v_2, v_3, v_4 and determinant 4.

Compute the determinant of the matrices

$(v_1, -30\, v_2,\ v_3, v_4)$

$(v_2, v_3,\ v_1, v_4)$

$(v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4)$

Algorithm: Using Gauss-Jordan to compute determinant

- Consider a square matrix A. Suppose that in the process of computing rref(A) one arrives to a matrix B by swaping rows s times and dividing columns by scalars $k_1, k_2,..,k_r$. Then

  $\det(A) = k_1 . K_2 \ldots k_r . \det(B)$.

In particular, if B=rref(A)

$$\det(A) = k_1 . K_2 \ldots k_r .$$

EXAMPLE 5.2-9

- Compute the determinant using gaussian elimination

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{vmatrix}$$

Theorem

- Consider row columns $v_1, v_2, .. v_{i-1}, v_{i+1}, .. v_n$ with n entries. The function from R^n to R,

$T(x) = \det(v_1\ v_2\ ..\ v_{i-1} x\ v_{i+1}\ ..\ v_n)$ is linear.

Theorem (for math curious students)

- The determinant is the only function from $(R^n)^n$ to R such that
  - It is linear on each rows (fixing the all the other rows).
  - It is alternating (swapping rows changes sign)
  - It has value 1 on the identity matrix.

Theorem

- If A is an n x n matrix with orthogonal columns v_1, v_2, …v_n then

$|\det A| = ||v_1|| . ||v_2|| . \dots ||v_n||$.

If B is an n x n matrix

$|\det B| = ||v_1|| . ||v_2^\perp|| . || \dots || v_n^\perp||$, where $v_i^\perp$ is the component of v_i perpendicular to span($v_1,\dots,v_{i-1}$).

Definition

- Consider vectors v_1 ,v_2 .... v_m in R^n. The m-parallelepiped defined by the vectors v_1 ,v_2 .... v_m is the set of all vectors of the form $c_1 v_1 + c_2 v_2 + \dots + c_m v_m$ where c_1 , c_2 ….c_m are scalars such that $0 \le c_i \le 1$.

Theorem

- Consider vectors v_1 ,v_2 .... v_m in R^n. The volume of the m-parallelepiped defined by v_1 ,v_2 .... v_m is $\sqrt{A^t.A}$ where A is the matrix with columns v_1 ,v_2 .... v_m .
- If m=n then the volume is |detA|.

Example

- Find the volume