MAT211 Lecture 16

Orthogonal transformations and orthogonal matrices

Definition

A linear transformation from R^n to R^n is called *orthogonal* if it preserves the length vectors. In symbols,

||T(x)|| = ||x|| for all x in \mathbb{R}^n .

The matrix A of an orthogonal transformation is said to be an *orthogonal matrix*.

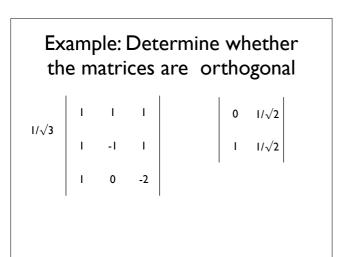
Two examples:

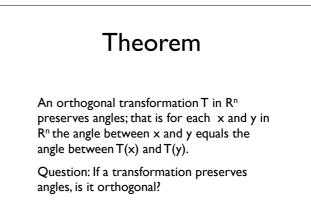
A rotation in R^2 and a reflexion in $R^{\mbox{\tiny n}}$ are orthogonal transformations.

Questions:

Are projections orthogonal transformations?

• What is the kernel of an orthogonal transformation?



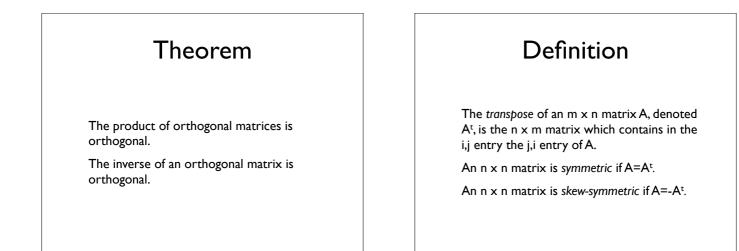


Theorem

A linear transformation from R^n to R^n is orthgonal if and only if the vectors $(T(e_1), T(e_2), ..., T(e_n))$ form an orthonormal basis.

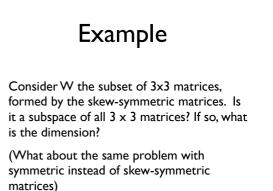
A matrix A is orthogonal if an only if the columns of A form an orthonormal basis.

Example: Determine whether the matrices are orthogonal



Example: Find A^{t.} Determine whether A is symmetric. Determine whether A is skew-symmetric

I	Ι	Ι	0	I/√2
I	-1	I	I	1/√2
I	I	2		I



Theorem

- If v and w are column vectors in Rⁿ then the dot product of v and w equals the matrix product v.w^t.
- An n x n matrix is orthogonal if and only if A.A^t = I_n.

Example: Determine whether the matrices are orthogonal

I /√3	I /√2	1/2	0	I/√2
I /√3	/√2	1/2	I	1/√2
I /√3	0	-1 /√2		

Theorem

- If A is an n x p matrix, and B is an p x m matrix then (A.B)^t=B^t.A^t.
- $\$ If A is an orthogonal matrix then A⁻¹=A^t.
- If a is an n x n invertible matrix, then A^t is also invertible and (A^t)⁻¹=(A⁻¹)^t.
- For any matrix A, rank(A)=rank(A^t)

Theorem

If $u_1, u_2,...u_m$ is an orthonornal basis of a subspace V of R^n then the matrix of the projection onto V is $Q.Q^t$ where Q is the matrix with columns $u_1, u_2,..u_m$

Review An n x n matrix is orthogonal if and only if $A.A^t = I_n$. A matrix is symmetric if A=A^t. A matrix is skew-symmetric if A=-A^t.

Example 5.3 35

Find orthogonal transformation T from R3 to such that $T(\sp{3},\sp{3},\sp{3}){=}(0,0,1)$

Find the matrix of the orthogonal projection of the line in \mathbb{R}^n spanned by the vector. (1,1...1)

Given an example of a non-zero skew symmetric matrix A and compute A^2

Let A be the matrix of an orthogonal projection. Find A^2 in two ways

Geometrically

Using the formula we saw.