MAT211 Lecture 15
Gram-Schmidt Process

* The Gram-Schmidt process
* QR Factorization

The Gram-Schmidt Process

**EXAMPLE**

Perform the Gram-Schmidt process on the sequence of vectors \((1,1,1), (2,0,2), (-1,0,-1)\)

**Theorem (QR Factorization)**

Consider an \(n \times n\) matrix \(M\) with linearly independent columns \(v_1, v_2, \ldots, v_n\).

Then there exists an \(n \times n\) matrix \(Q\) whose columns \(u_1, u_2, \ldots, u_n\) are orthonormal and an upper triangular matrix \(R\) with positive diagonal entries such that \(M = QR\).

The matrices \(Q\) and \(R\) are unique with the above properties. Moreover, \(r_{11} = |v_1|\), \(r_{jj} = |v_j|\) for \(j = 2, \ldots, n\), and \(r_{ij} = u_i \cdot v_j\) for \(i < j\).

**Theorem (QR Factorization Algorithm)**

Consider an \(n \times n\) matrix \(M\) with linearly independent columns \(v_1, v_2, \ldots, v_n\).

Then the columns \(q_1, q_2, \ldots, q_n\) of \(Q\) and the columns of \(R\) can be computed in the following order:

- First col of \(R\), first column of \(U\)
- Second col of \(R\), second col of \(U\)
- and so on

**EXAMPLE: Find the QR factorization of the matrix**

\[
\begin{pmatrix}
1 & 2 & 1 \\
1 & 1 & 1 \\
1 & 0 & 2 \\
1 & 1 & 1 \\
\end{pmatrix}
\]