MAT211 Lecture 14

Orthogonal projections and orthogonal basis

- Orthogonality, length, unit vectors
- Orthonormal vectors: definition and properties
- Orthogonal projections: definition, formula and properties.
- Orthogonal complements
- Pythagorean theorem, Cauchy inequality, angle between two vectors

Definition

- Two vectors u and v in R^n are *perpendicular or orthogonal* if u.v=0
- The *length* of a vector v in R^n is $||v||=\sqrt{v.v}$
- A vector v in R^n is called a *unit vector* if $||v||=1$

Example

- Find a unit vector in the line of multiples of (1,1,3)
- Find a vector of length 2 orthogonal to (1,1,3)

Definition

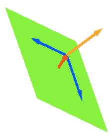
A vector v in R^n is orthogonal to a subspace V of R^n if it is orthogonal to all vectors in V

Remark: If $(b_1,b_2,..,b_m)$ is a basis of V, then v is orthogonal to V if (and only if) v is orthogonal to $b_1,b_2,..$ and b_m.

EXAMPLE

Consider the subspace V of R^3 span by (1,1,1) and (1,0,1).

Find all the vectors orthogonal to V.



Definition

The vectors u_1, u_2, .. u_m of R^n are called *orthonormal* if they are all unit vectors and are orthogonal to one another. In symbols

$(u_i,u_j)=0$ if $i \neq j$

$(u_i,u_i)=1$

EXAMPLE (5.2-33)

Find an orthonormal basis of the kernel of the matrix

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$$

EXAMPLE

Consider the vectors $v_1=(1/2)(1,1,1,1)$, $v_2=(1/\sqrt{2})(1,1,1,1)$, $v_3=(1/2)(1,-1,1,-1)$. Find a vector v_4 such that v_1, v_2, v_3, v_4 form an orthonormal basis of R^4

Theorem

- Orthonormal vectors are linearly independent.
- A set of n orthonormal vectors in R^n form a basis.

EXAMPLE:

Consider the vectors $v_1=(1/2)(1,1,1,1)$, $v_2=(1/\sqrt{2})(1,1,1,1)$, $v_3=(1/2)(1,-1,1,-1)$. Find a vector v_4 such that v_1, v_2, v_3, v_4 form an orthonormal basis of R^4

Write $(1,0,0,0)$ as a linear combination of v_1, v_2, v_3, v_4

Theorem

Let V be a subspace of R^n and let x be a vector R^n. Then there exists unique vectors $x^{\perp}$ and $x^{||}$ such that

- $x = x^{||} + x^{\perp}$
- $x^{||}$ in V
- $x^{\perp}$ is orthogonal to V.

Theorem

If V us a subspace of R^n with orthonormal basis $(b_1,b_2,..,b_m)$ then

$$\boxed{proj_V(x)=(b_1.x)\, b_1 + (b_2.x)\, b_2 + \ldots (b_m.x)\, b_m}$$

In particular if $V= R^n$

$$\boxed{x=(b_1.x)\, b_1 + (b_2.x)\, b_2 + \ldots (b_n.x)\, b_n}$$

Example

Find the orthogonal projection of (1,2,3) onto the subspace of R^3 spanned by (1,1,0) and (1,0,0).

Definition

Consider V a subspace of R^n. The orthogonal complement $V^\perp$ of V is the set of vectors x of R^n that are orthogonal to all vectors in V.

In other words $V^\perp$ is the kernel of the linear transformation $proj_V$

Theorem: Consider V, a subspace of R^n

- The orthogonal complement of V is a subspace of R^n
- $V \cap V^\perp = \{0\}$
- $\dim(V)+\dim(V^\perp)=n$
- $(V^\perp)^\perp=V$

Example

Find the orthogonal complement V where V is the subspace of R^3 spanned by (1,1,0) and (1,0,0).

Theorem: Consider two vectors x and y

- $||x+y||^2=||x||^2 +||y||^2$ if an only if x and y are orthogonal (Pythagorean theorem)
- If V is a subspace of R^n then $||proj_V(x)||\leq||x||$
- Cauchy-Schwarz Inequality: $|x.y|\leq||x||.||y||$.

Definition

Consider two non-zero vectors x and y in R^n. The angle θ between these two vectors is defined as arc $\cos(x.y/||x||.||y||)$.

EXAMPLE

Find the angle between the vectors x=
(1,1,1) and (1,0,1).