MAT211 Lecture 14

Orthogonal projections and orthogonal basis
- Orthogonality, length, unit vectors
- Orthonormal vectors: definition and properties
- Orthogonal projections: definition, formula and properties.
- Orthogonal complements
- Pythagorean theorem, Cauchy inequality, angle between two vectors

Example

- Find a unit vector in the line of multiples of \((1,1,3)\)
- Find a vector of length 2 orthogonal to \((1,1,3)\)

Definition

- Two vectors \(u\) and \(v\) in \(\mathbb{R}^n\) are **perpendicular or orthogonal** if \(u \cdot v = 0\)
- The **length** of a vector \(v\) in \(\mathbb{R}^n\) is \(\|v\| = \sqrt{v \cdot v}\)
- A vector \(v\) in \(\mathbb{R}^n\) is called a **unit vector** if \(\|v\| = 1\)

Example

Consider the subspace \(V\) of \(\mathbb{R}^3\) span by \((1,1,1)\) and \((1,0,1)\).
Find all the vectors orthogonal to \(V\).

Definition

- The vectors \(u_1, u_2, \ldots, u_m\) of \(\mathbb{R}^n\) are called **orthonormal** if they are all unit vectors and are orthogonal to one another. In symbols

\[
(u_i \cdot u_j) = 0 \quad \text{if} \quad i \neq j
\]
\[
(u_i \cdot u_i) = 1
\]
EXAMPLE (5.2-33)

Find an orthonormal basis of the kernel of the matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]

EXAMPLE

Consider the vectors \(v_1=(1/2)(1,1,1,1)\), 
\(v_2=(1/\sqrt{2})(1,1,1,1)\), 
\(v_3=(1/2)(1,-1,1,-1)\). Find a vector \(v_4\) such that \(v_1, v_2, v_3, v_4\) form an orthonormal basis of \(\mathbb{R}^4\)

Theorem

- Orthonormal vectors are linearly independent.
- A set of \(n\) orthonormal vectors in \(\mathbb{R}^n\) form a basis.

EXAMPLE:

Consider the vectors \(v_1=(1/2)(1,1,1,1)\), 
\(v_2=(1/\sqrt{2})(1,1,1,1)\), 
\(v_3=(1/2)(1,-1,1,-1)\). Find a vector \(v_4\) such that \(v_1, v_2, v_3, v_4\) form an orthonormal basis of \(\mathbb{R}^4\)
Write \((1,0,0,0)\) as a linear combination of \(v_1, v_2, v_3, v_4\)

Theorem

Let \(V\) be a subspace of \(\mathbb{R}^n\) and let \(x\) be a vector \(\mathbb{R}^n\). Then there exists unique vectors \(x\parallel\parallel\) and \(x\perp\) such that

- \(x = x\parallel\parallel + x\perp\)
- \(x\parallel\parallel\) in \(V\)
- \(x\perp\) is orthogonal to \(V\).

Theorem

If \(V\) is a subspace of \(\mathbb{R}^n\) with orthonormal basis \((b_1, b_2, \ldots, b_n)\) then

\[
\text{proj}_V(x) = (b_1, x) b_1 + (b_2, x) b_2 + \ldots + (b_n, x) b_n
\]

In particular if \(V = \mathbb{R}^n\)

\[
x = (b_1, x) b_1 + (b_2, x) b_2 + \ldots + (b_n, x) b_n
\]
**Example**

Find the orthogonal projection of $(1,2,3)$ onto the subspace of $\mathbb{R}^3$ spanned by $(1,1,0)$ and $(1,0,0)$.

**Definition**

Consider $V$ a subspace of $\mathbb{R}^n$. The orthogonal complement $V^\perp$ of $V$ is the set of vectors $x$ of $\mathbb{R}^n$ that are orthogonal to all vectors in $V$.

In other words $V^\perp$ is the kernel of the linear transformation $\text{proj}_V$.

**Theorem: Consider $V$, a subspace of $\mathbb{R}^n$**

- The orthogonal complement of $V$ is a subspace of $\mathbb{R}^n$.
- $V \cap V^\perp = \{0\}$
- $\dim(V) + \dim(V^\perp) = n$
- $(V^\perp)^\perp = V$

**Example**

Find the orthogonal complement $V$ where $V$ is the subspace of $\mathbb{R}^3$ spanned by $(1,1,0)$ and $(1,0,0)$.

**Theorem: Consider two vectors $x$ and $y$**

- $||x+y||^2 = ||x||^2 + ||y||^2$ if and only if $x$ and $y$ are orthogonal (Pythagorean theorem)
- If $V$ is a subspace of $\mathbb{R}^n$ then $||\text{proj}_V(x)|| \leq ||x||$.
- Cauchy-Schwarz Inequality: $|x \cdot y| \leq ||x|| \cdot ||y||$.

**Definition**

Consider two non-zero vectors $x$ and $y$ in $\mathbb{R}^n$. The angle $\theta$ between these two vectors is defined as $\arccos(x \cdot y / ||x|| \cdot ||y||)$. 
EXAMPLE

Find the angle between the vectors \( x = (1,1,1) \) and \( (1,0,1) \).