MAT211 Lecture 13
The matrix of a linear transformation
✦ The B-matrix of a linear transformation
✦ The columns of the B-matrix of a linear transformation
✦ Change of basis matrix
✦ Change of basis in a subspace of \( R^n \)
✦ Change of basis for the matrix of a linear transformation

Overview
- \( A \) and \( B \) basis of linear space \( V \), \( T \) a linear transformation from \( V \) to \( V \).
- Coordinate Transformation from \( V \) to \( R^n \) \( L(f) = [f]_B \)
- B-matrix of \( T \) is \( L_B \circ T \circ L_B^{-1} \)
- Change of basis from \( B \) to \( A \), \( S_{B \rightarrow A} = L_A \circ (L_B)^{-1} \)
- If \( B \) is \( B \)-matrix of \( T \) and \( A \) is \( A \)-matrix of \( T \), \( S \) the change of basis from \( B \) to \( A \), \( AS = S B \)

EXAMPLE
- Consider the space \( U \) of upper triangular \( 2 \times 2 \) matrices and the linear transformation \( T \) from \( U \) to \( U \) defined by \( T(M) = AM \) where \( A \) is
  \[
  \begin{pmatrix}
  1 & -2 \\
  0 & 3
  \end{pmatrix}
  \]
  For each element \( z \) of \( U \), find \( [T(z)]_B \) where \( B \) is the standard basis

Definition
- Consider a linear transformation \( T \) from \( V \) to \( V \) where \( V \) is an \( n \)-dimensional linear space. Let \( B \) denote a basis of \( V \).
- The matrix \( B \) of the transformation from \( R^n \) to \( R^n \) defined by \( L_B \circ T \circ L_B^{-1} \) is called the \( B \)-matrix of \( T \).

EXAMPLE
- Consider the space \( U \) of upper triangular \( 2 \times 2 \) matrices and the linear transformation \( T \) from \( U \) to \( U \) defined by \( T(M) = AM \) where \( A \) is
  \[
  \begin{pmatrix}
  1 & .2 \\
  0 & 3
  \end{pmatrix}
  \]
  Find the \( B \)-matrix of \( T \) where \( B \) is the standard basis of \( U \).

Theorem
- Consider a linear transformation \( T \) from \( V \) to \( V \). Let \( B \) be matrix of \( T \) with respect to a basis \( B = (b_1, b_2, \ldots, b_n) \)
- Then columns of \( B \) are the \( B \)-coordinate vectors \( [T(b_1)]_B, [T(b_2)]_B, \ldots, [T(b_n)]_B \)
EXAMPLE

• Give the matrix of the linear transformation \(T(f) = f'' + 2f\) from \(P_2\) to \(P_2\) with respect to the basis \((1,t,t^2)\).
• Find basis of the kernel and the image and compute rank and nullity of \(T\).

Definition

• Consider two bases \(A\) and \(B\) of an \(n\)-dimensional vector space \(V\).
• The matrix \(S = S_{B \rightarrow A}\) of the linear transformation \(L_A \circ (L_B)^{-1}\)
• \((L_B)^{-1}\) from \(R^n\) to \(R^n\) is called the change of basis matrix from \(B\) to \(A\).

EXAMPLE

• Find the change of basis matrix \(S\) from the standard basis \(B\) to basis \(A\) of \(U^{2x2}\) where \(A\) is

\[
\begin{pmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Find the change of basis matrix from \(A\) to \(B\).

Remarks

• If \(B= (b_1, b_2, ... b_m)\) then the columns of \(S\) are \(S_1, S_2, ..., S_m\) and

\[S_1 = [b_1]_A, S_2 = [b_2]_A, ..., S_m = [b_m]_A\]

Remarks

• If \(A\) and \(B\) are two basis of a vector space then \(S_{B \rightarrow A}\) is an invertible matrix and

\[S_{B \rightarrow A} = (S_{A \rightarrow B})^{-1}\]

Theorem

• Consider a subspace \(V\) of \(R^n\) and two basis of \(V\), \(A = (a_1, a_2, ... a_m)\) and \(B = (b_1, b_2, ... b_m)\). Denote by \(S\) the change of basis matrix from \(B\) to \(A\).
• Then \([b_1, b_2, ... b_m] = [a_1, a_2, ... a_m] S\).
EXAMPLE (4.3-60)

- In the plane $V$ defined by the equation $2x+y-2z=0$ consider the basis
  \[ A=\{(1,2,2),(2,-2,1)\} \]
  \[ B=\{(1,2,2),(3,0,3)\} \]

1. Find the change of basis matrix from $B$ to $A$
2. Find the change of basis matrix from $A$ to $B$
3. Write an equation relating the matrices $[a_1,a_2]$ and $[b_1,b_2]$ where $A=\{(a_1,a_2)\}$ and $B=\{(b_1,b_2)\}$

Theorem

- Consider a linear space $V$ and two basis of $V$, $A=(a_1, a_2, \ldots, a_n)$ and $B=(b_1, b_2, \ldots, b_n)$. Let $T$ be a linear transformation from $V$ to $V$, and let $A$ and $B$ be the $A$-matrix and the $B$-matrix of $T$, respectively. Then $A$ is similar to $B$ and
  \[ AS=SB \]
  where $S$ is the change of basis matrix from $B$ to $A$.

EXAMPLE

- Find the change of basis matrix $S$ from the standard basis $B$ of $\mathbb{R}^{2x2}$ to basis $A$ where $A$ is
  \[
  \begin{bmatrix}
  1 & 0 & 0 & 1 \\
  0 & 1 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
  \end{bmatrix}
  \]
  \[
  \begin{bmatrix}
  1 & -2 \\
  0 & 3
  \end{bmatrix}
  \]

Let $T$ be the transformation $T(M)=AM$ where $A$ is

Verify the formula $SB = AS$ (A is the $A$-matrix of $T$, $B$ is the $B$-matrix of $T$)

Find the change of basis matrix from $A$ to $B$.

$A$ and $B$ basis of linear space $V$, $T$ a linear transformation from $V$ to $V$.

- Coordinate Transformation from $V$ to $\mathbb{R}^n L(f) = [f]_B$
  \[
  B\text{-}matrix \ of \ T \ is \ L_B \ o \ T \ o \ L_B^{-1}
  \]
- Change of basis from $B$ to $A$, $S_{B\rightarrow A}=L_A \ o \ (L_B)^{-1}$
- If $B$ is $B$-matrix of $T$ and $A$ is $A$-matrix of $T$, $S_{B\rightarrow A}=S_{A\rightarrow B} \ B$