MAT211 Lecture 13

The matrix of a linear transformation

✦The B-matrix of a linear transformation
✦ The columns of the B-matrix of a linear transformation
✦Change of basis matrix
✦Change of basis in a subspace of R^n
✦Change of basis for the matrix of a linear transformation

Overview

A and **B** basis of linear space V, T a linear transformation from V to V.

- Coordinate Transformation from V to R^n $L(f)=[f]_B$
- **B**-matrix of T is $L_{\mathbf{B}} \circ T \circ L_{\mathbf{B}}^{-1}$
- Change of basis from B to A, $S_{\mathbf{B}\to\mathbf{A}}=L_{\mathbf{A}} \circ (L_{\mathbf{B}})^{-1}$
- If B is **B**-matrix of T and A is **A**-matrix of T, S the change of basis from B to A, AS=S B

EXAMPLE

- Consider the space U of upper triangular 2 x 2 matrices and the linear transformation T from U to U defined by T(M)=AM where A is
$$\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}$$
For each element z of U, find $[T(z)]_B$,where B is the standard basis

Definition

- Consider a linear transformation T from V to V where V is an n-dimensional linear space. Let **B** denote a basis of V.
- The matrix B of the transformation from R^n to R^n defined by $L_{\mathbf{B}} \circ T \circ L_{\mathbf{B}}^{-1}$ is called the ***B****-matrix of T.*

EXAMPLE

- Consider the space U of upper triangular 2 x 2 matrices and the linear transformation T from U to U defined by T(M)=AM where A is
$$\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}$$
Find the **B**-matrix of T where **B** is the standard basis of U.

Theorem

- Consider a linear transformation T from V to V. Let B be matrix of T with respect to a basis $\mathbf{B}=(b_1,b_2,..,b_n)$
- Then columns of B are the B-coordinate vectors $[T(b_1)]_{\mathbf{B}}$,$T(b_2)]_{\mathbf{B}}$,..,$T(b_n)]_{\mathbf{B}}$

EXAMPLE

- Give the matrix of the linear transformation $T(f)=f''+2f'$ from P_2 to P_2 with respect to the basis $(1,t,t^2)$.
- Find basis of the kernel and the image and compute rank and nullity of T.

Definition

- Consider two basis **A** and **B** of an n-dimensional vector space V.
- The matrix $S = S_{\mathbf{B}\to\mathbf{A}}$ of the linear transformation
- $L_{\mathbf{A}} \circ (L_{\mathbf{B}})^{-1}$
- from R^n to R^n is called the *change of basis matrix from B to A.*

EXAMPLE

- Find the change of basis matrix S from the standard basis **B** to basis **A** of U^{2x2} where **A** **is**

$$\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$$

Find the change of basis matrix from **A** to **B**.

Remarks

- If f is in V then $[f]_{\mathbf{A}} = S\,[f]_{\mathbf{B}}$ where S is the change of basis matrix from **B** to **A**
- If $\mathbf{B}=(b_1,b_2,..,b_n)$ then the columns of S are $Se_1, Se_2,.., Se_n$ and

$Se_1=[b_1]_{\mathbf{A}}, Se_2=[b_2]_{\mathbf{A}}, \ldots, Se_n=[b_n]_{\mathbf{A}}$

Remarks

- If A and B are two basis of a vector space then $S_{\mathbf{B}\to\mathbf{A}}$ is an invertible matrix and
- $S_{\mathbf{B}\to\mathbf{A}} = (S_{\mathbf{A}\to\mathbf{B}})^{-1}$

Theorem

- Consider a subspace V of R^n and two basis of V, $\mathbf{A}=(a_1,a_2,..,a_m)$ and $\mathbf{B}=(b_1,b_2,..,b_m)$. Denote by S the change of basis matrix from **B** to **A**.
- Then $[b_1,b_2,..,b_m] = [a_1,a_2,..,a_m]S$.

EXAMPLE (4.3-60)

- In the plane V defined by the equation 2x+y-2z=0 consider the basis
- A=((1,2,2),(2,-2,1)) and B=((1,2,2),(3,0,3))

1. Find the change of basis matrix from B to A
2. Find the change of basis matrix from A to B
3. Write an equation relating the matrices $[a_1,a_2]$ and $[b_1,b_2]$ where A= (a_1,a_2) and B= (b_1,b_2)

13

Theorem

- Consider a linear space V and two basis of V, **A**=$(a_1,a_2,..,a_n)$ and **B**=$(b_1,b_2,..,b_n)$. Let T be a linear transformation from V to V, and let A and B be the **A**-matrix and the **B**-matrix of T, respectively. Then A is similar to B and
- AS=SB
- where S is the change of basis matrix from **B** to **A.**

14

EXAMPLE

- Find the change of basis matrix S from the standard basis **B** of U^{2x2} to basis **A** where **A** is $\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$

Let T be the transformation T(M)=AM where A is $\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}$

Verify the formula SB = AS (A is the **A**-matrix of T, B is the **B**-matrix of T)

Find the change of basis matrix from **A** to **B**.

15

A and **B** basis of linear space V, T a linear transformation from V to V.

- Coordinate Transformation from V to R^n L(f)= $[f]_B$
- **B**-matrix of T is $L_{\mathbf{B}}$ o T o $L_{\mathbf{B}}^{-1}$
- Change of basis from B to A, $S_{\mathbf{B->A}}$=$L_{\mathbf{A}}$ o $(L_{\mathbf{B}})^{-1}$
- If B is **B**-matrix of T and A is **A**-matrix of T, $S_{\mathbf{B->A}}$ A=$S_{\mathbf{A->B}}$ B