MAT131- Midterm 1
February 27, 2003

Instructions: You have 90 minutes to answer all six questions. You are not allowed to use any books or notes, but you may use your calculator. Show all the work you want to be graded. Write carefully, points may be taken off for meaningless statements. Good luck.

1				2			3				
a	b	c	d	a	b	c	a	b	c	d	e
/5	/5	/5	/5	/5	/5	/5	/3	/3	/4	/4	/5

4		5					6				Total
a	b	a	b	c	d	e	a	b	c	d	
/5	/5	/6	/3	/3	/3	/5	/4	/4	/4	/4	

NOTE: SOME PROBLEMS HARD PROBLEMS WILL GIVE EXTRA CREDIT.

SO, THIS ANSWER KEY ADDS UP TO MORE THAN 100

(1) Evaluate the following limits if they exist, and if not, explain why not.

(a) $\lim_{x\to 0}\cos(\pi e^{2x})$: Since $f(x) = \pi e^{2x}$ and $g(x) = \cos x$ are continuous functions at all real numbers, the composition $g(f(x)) = \cos(\pi e^{2x}$ is continuous at all real numbers Therefore,

$\lim_{x\to 0}\cos(\pi e^{2x}) \overset{Justification}{=} \cos(\pi e^{2\cdot 0}) = \cos(\pi e^0 = \cos(\pi) = -1$

(b) $\lim_{t\to 12}\frac{2-\sqrt{t-8}}{t-12} = \lim_{t\to 12}\frac{(2-\sqrt{t-8})(2+\sqrt{t-8})}{(t-12)(2+\sqrt{t-8})} = \lim_{t\to 12}\frac{(2-(t-8)}{(t-12)(2+\sqrt{t-8})} = \lim_{t\to 12}\frac{1}{(2+\sqrt{t-8})} \overset{Justification}{=} \frac{-1}{4}$

Since $f(t) = 2+\sqrt{t-8}$ is continuous at $x = 12$ and $f(12) \neq 0$, then $\frac{1}{f(t)} = \frac{1}{(2+\sqrt{t-8})}$ is a continuous function at $x = 12$

$\lim_{x\to -1} \frac{x^3+2x^2+x}{(x+1)^2} = \lim_{x\to -1} \frac{x(x+1)^2}{(x+1)^2} = \lim_{x\to -1} \frac{x}{1} = -1$
$f(x) = x$ is a continuous function at all the real numbers.

(c) $\lim_{x\to -1} \frac{x^2-1}{(x+1)^2} = \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)^2} = \lim_{x\to -1} \frac{x-1}{(x+1)}$.
The limit does not exist, because close -1 the numerator is close to -1 (and so, different from 0) and the denominator is close to 0. Moreover, $\lim_{x\to -1} \frac{x^2-1}{(x+1)^2} = \infty$ $Lim = number \neq 0/0$ is indetermined is also accepted as good answer

(2) (a) Find the interval below which contains a solution of the equation $\log_2(2x) - x^2 + 5 = 0$. Justify your answer.

(i) $[\frac{1}{2}, 1]$
(ii) $[1, 2]$
(iii) $[2, 4]$

The function $f(x) = \log_2(2x) - x^2 + 5$ is continuous at all positive numbers. Therefore, one can apply intermediate value theorem.

One studies the value of the function at the endpoints of the intervals.

(i) $f(\frac{1}{2}) = \log_2 1 - 1 + 5 = 4$; $f(1) = \log_2(2) - 1 + 5 = 5$. Since the values of f at both endpoints are positive we cannot conclude that there is a solution in that interval.
(ii) $f(1) = 5$, $f(2) = \log_2 4 - 4 + 5 = 3$. As in (i), we cannot conclude that there is a solution in that interval.
(iii) $f(2) = 3$, $f(4) = log_2 8 - 16 + 5 = 3 - 16 + 5 = -8$. Since $f(2) > 0$ and $f(4) > 0$, there exists a number c in $(2, 4)$ such that $f(c) = 0$.

(b) Find an interval of length 1 that contains a solution of the equation $2x^3 - x^2 - 3x + 1 = 0$

Let $f(x) = 2x^3 - x^2 - 3x + 1$. Then $f(0) = 1$ and $f(1) = -1$. Since $f(0) > 0$, $f(1) < 0$ and $f(x) = 2x^3 - x^2 - 3x + 1$ is a continuous function at all the real numbers (because it is a polynomial) then we can apply the intermediate value theorem to conclude that there is number c between 0 and 1 such that $f(c) = 0$. The interval $[0, 1]$ is a solution (not the only one, each of the intervals $[-2, -1]$) and $[1, 2]$ alson contain a solution of the equation $f(x) = 0$

(c) Find an interval that contains a solution of the equation $\cos x = x$. (Intervals with endpoints equal to ∞ of $-\infty$ are not allowed)

Consider the function $f(x) = \cos x - x$. Since $f(0) = 1 > 0$ and $f(\pi) = 1 - \pi < 0$ and $f(x)$ is a continuous function at all the real numbers, we can apply the intermediate value theorem to conclude that there is number c between 0 and π such that $f(c) = 0$.

(3) Consider a real number c and a function $f(x)$ depending on the parameter c.

$$f(x) = \begin{cases} x^5 & \text{if } x < 1. \\ x & \text{if } 1 \leq x \leq 3. \\ cx^2 & \text{if } x > 3 \end{cases}$$

(a) Sketch a graph of $y = f(x)$ in each of the following cases.

(i) $c = 1$

(ii) $c = 0$

$\boxed{\text{Solution: See graph at the end}}$ 1.5 points each graph, 0.5 each "piece" of function.

(b) Find $\lim_{x \to -2} f(x)$

Since at $x = 2$, $f(x)$ is a continuous function and $-2 < 1$, $\lim_{x \to -2} f(x) = f(-2) = (-2)^5 = -32$

(c) Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$

$\lim_{x \to 1^+} f(x) = 1$ and $\lim_{x \to 1^-} f(x) = 1$ because $f(x)$ is continuous according to the graph.

(d) Is $f(x)$ continuous at x=1? Yes, it is, because by the previous exercise, $\lim_{x \to 1} f(x) = f(1)$.

(e) Does there exist any value of c which makes the function f(x) continuous everywhere?

Since $\lim_{x \to 3^+} f(x) = 3$ $\lim_{x \to 3^-} f(x) = 9c$, the function $f(x)$ is continuous if and only if $3 = 9c$, that is, $c = 1/3$.

(4) The distance in miles from the departure point traveled by a car after t hours is given by d(t)

(a) Give a formula in terms of $d(1), d(2), d(3), \ldots$ for the average velocity during the third hour of travel.

Answer: $\frac{d(3)-d(2)}{3-2} = d(3) - d(2)$

(b) State in terms of the graph of $y = d(t)$ what is the instantaneous velocity at t=2.

Answer: The instantaneous velocity is the slope of the tangent line to the graph of $y = d(t)$ at the point $(2, d(2))$.

Other possible answer: $\lim_{t\to 2} \frac{d(t)-d(2)}{t-2}$

(5) (a) Sketch a graph of $y = f(x)$ if you know that $f(x)$ is a function with the following properties. (You may use the grid in this page to sketch the graph)

(i) $f(x)$ is a one-to-one function.

(ii) The domain of f(x) is the interval [-3,1]

(iii) The function f(x) is continuous in the intervals [-3,-1) and (-1,1] and it is not continuous at x=-1.

(iv) $f(-2) = -1.5$

(v) $f^{-1}(0.5) = -1$

Which, if any, of the following limits exists? Which limits, if any, can you find using what you know about $f(x)$. Justify your answers.

(b) $\lim_{x\to -2} f(x)$

(1) This limit exists because the function is continuous at $x = -2$,.

(2) $\lim_{x\to -2} f(x) = f(-2) = 1.5$ because the function is continuous.

(c) $\lim_{x\to -2^+} f(x)$

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(2) $\lim_{x\to -2^+} f(x) = f(-2) = 1.5$ because the function is continuous

(d) $\lim_{x\to -1^+} f(x)$

(1) One cannot determine whether the limit exist

(2) If it does exist, one cannot determine the value (it has to be different from $f(-1)$

(e) $\lim_{x \to -1} f(x)$

(1) One cannot determine whether the limit exist.

(2) If it does exist, one cannot determine the value (it has to be different from $f(-1)$

(6) In each case, state the interval or intervals where each of the following functions is continuous

For grading: All known notations for intervals are accepted.

(a) $f(x) = |x|$ Interval $(-\infty, \infty)$

(b) $f(x) = |x^7 - 10x^4 + 27|$ Interval $(-\infty, \infty)$ (because it is composition of two functions which are continuous on all real numbers)

(c) $\frac{x^4-4x^3+3x^2+3x+23}{x^2+4x+3}$ This is a continuous function for all the values of x for which $x^2 + 4x + 3 \neq 0$ That is, it is continuous if $x \neq -1$ and $x \neq -3$.The intervals are $(-\infty, -3)$, $(-3, -1)$ and $(-1, \infty)$

(d) $\ln(x^2 - 1)$ This function is continuous for all the values of x such that $x^2 - 1 > 0$. is, if for the values of x such that $x > 1$ or $x < -1$. The intervals are $(-\infty, -1)$ and $(1, \infty)$.

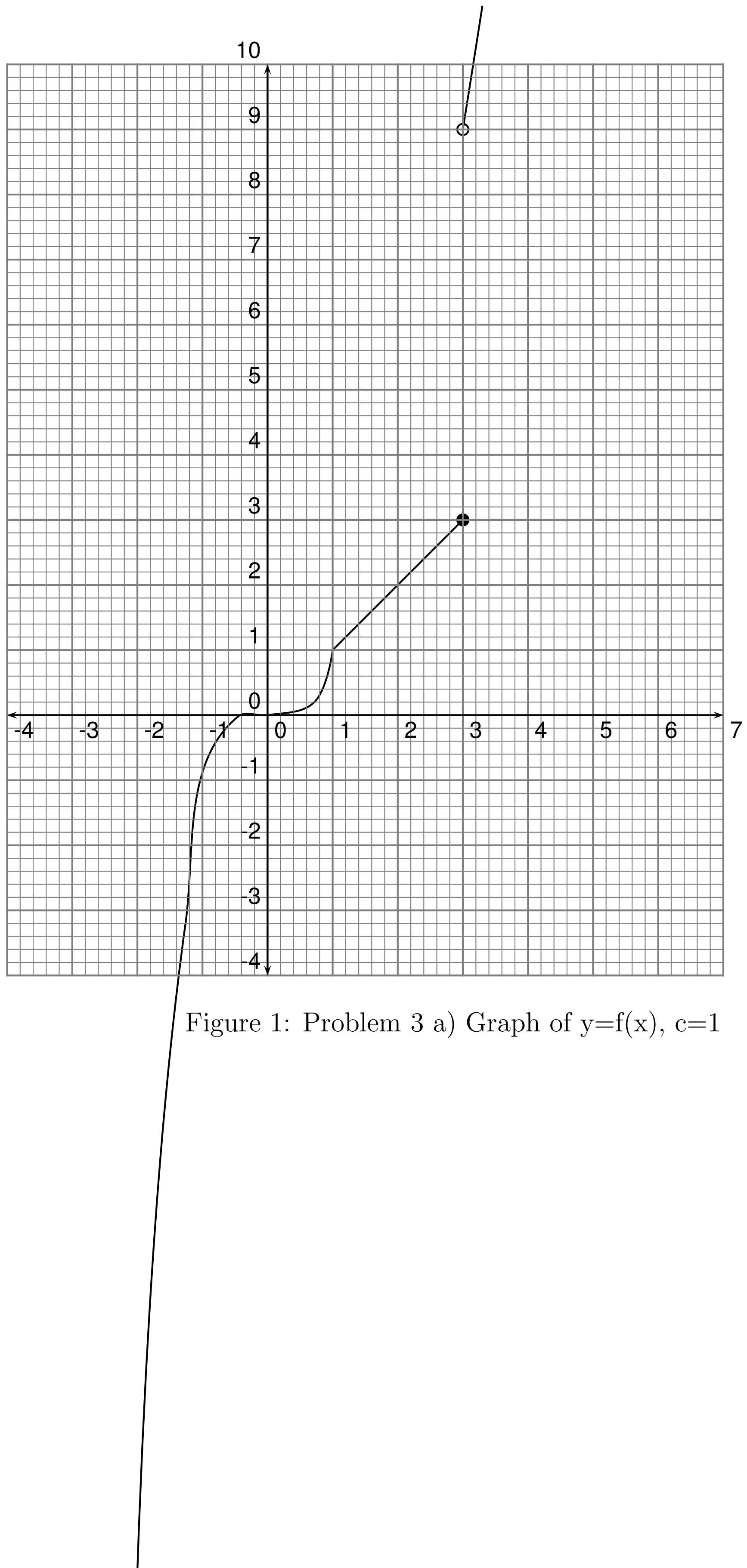

Figure 1: Problem 3 a) Graph of y=f(x), c=1

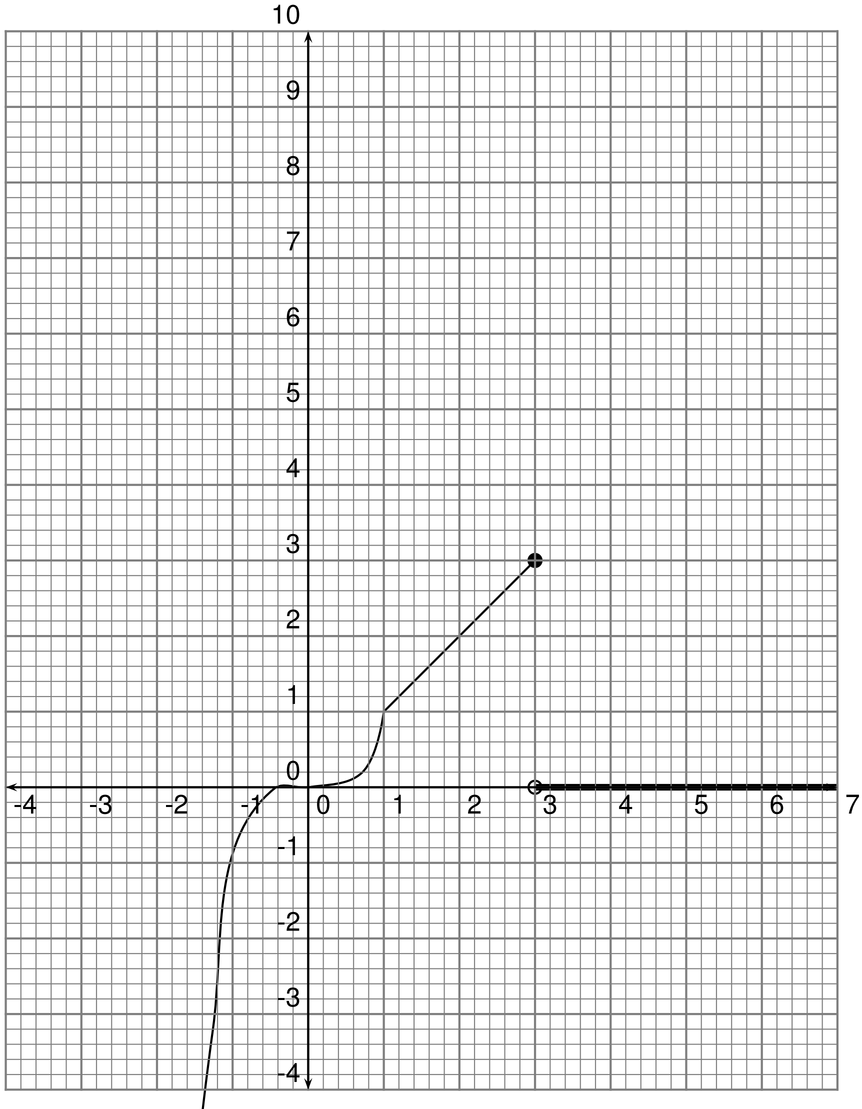

Figure 2: Problem 3 a) Graph of y=f(x), c=0

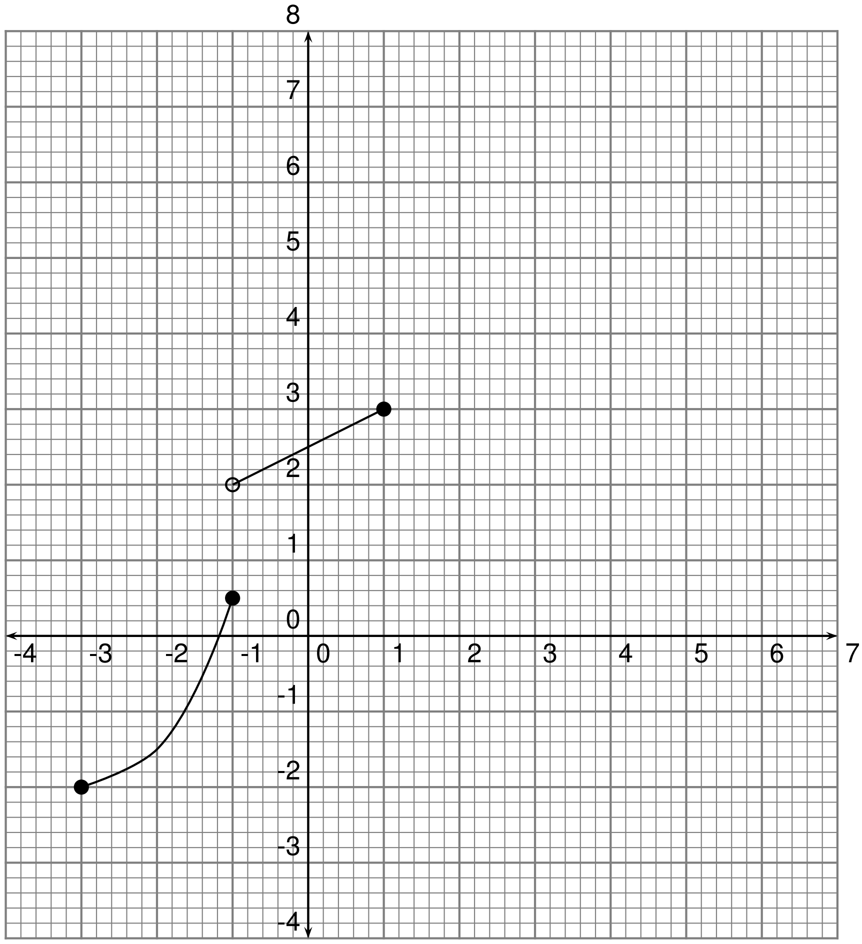

Figure 3: Problem 5, one of the possible graphs (not the only one)